Efficient calculation of covariances for astrometric data in the Gaia Catalogue

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gaia

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What kind of covariances do we expect between astrometric parameters?

How should we use the covariances?

How can this be done efficiently?

Covariance & correlation

x_i and x_j are estimated values of astrometric parameter i and j.

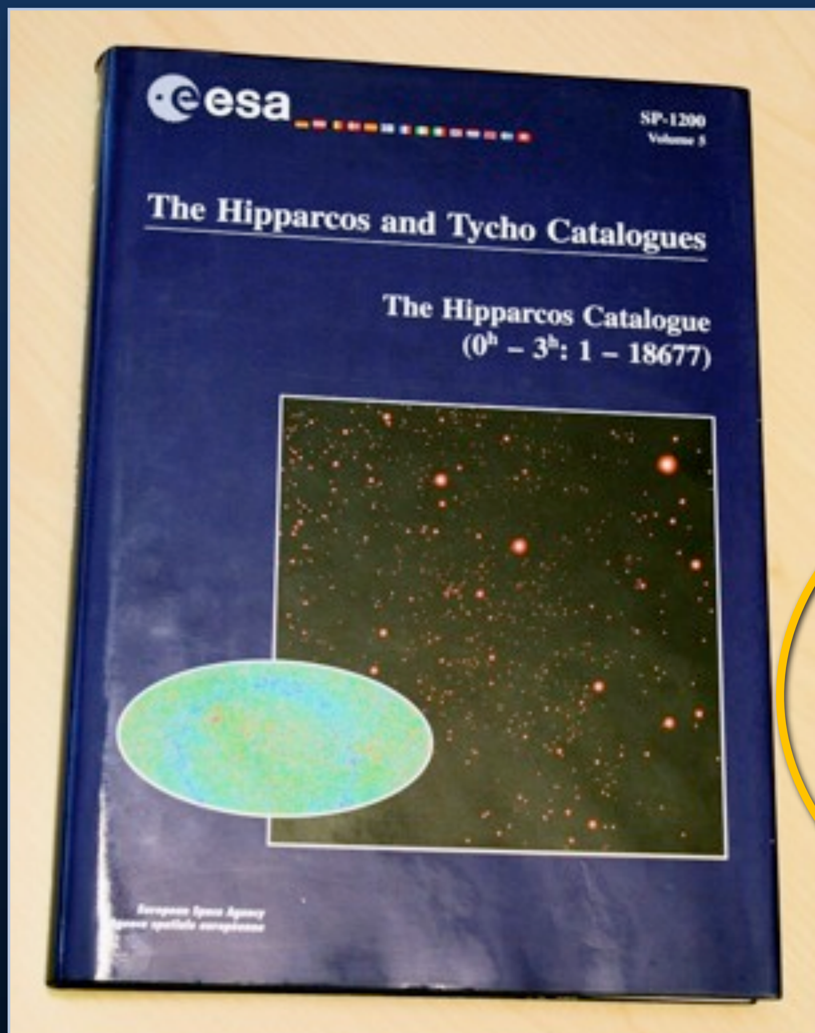
$$\begin{aligned} Cov[x_i, x_j] &= E[e_i e_j] \\ &= \rho_{ij}\sigma_i\sigma_j \\ \rho_{ij} &= \frac{E[e_i e_j]}{\sqrt{E[e_i^2]E[e_j^2]}} \end{aligned}$$

$e_i = (x_i - x_{i,true})$

$(= \sigma_i^2 \text{ for } i = j)$

correlation coefficient

Hipparcos catalogue



5 astrometric parameters

- estimated values and their standard errors

$$\alpha \;\; \delta \;\; \varpi \;\; \mu_{\alpha*} \;\; \mu_\delta$$

$$\sigma_\alpha \;\; \sigma_\delta \;\; \sigma_\varpi \;\; \sigma_{\mu_{\alpha*}} \;\; \sigma_{\mu_\delta}$$

- Correlation between astrometric parameters of each star

all 10 combinations

Covariances

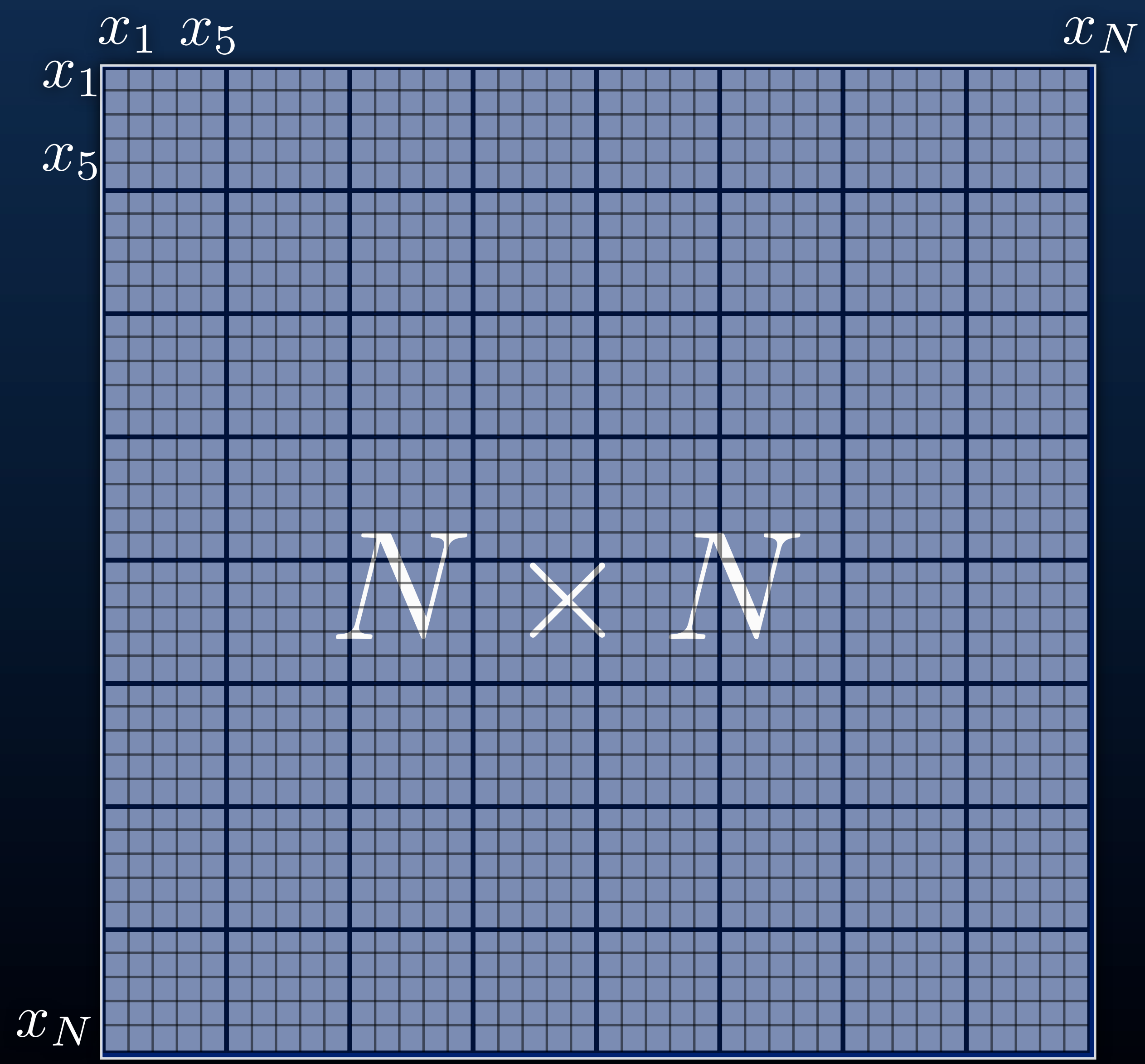
Number HIP (1)	(2)	Descriptor: epoch J1991.25 RA h m s (3)	Dec ±° ′ ″ (4)	V mag (5)	(6)	(7)	Position: epoch J1991.25 (ICRS) α deg (8)	δ deg (9)	(10)	Par. π mas (11)	Proper Motion $\mu_{\alpha*}$ mas/yr (12)	μ_δ mas/yr (13)
1		00 00 00.22	+01 05 20.4	9.10		H	0.000 911 85	+01.089 013 32		3.54	−5.20	−1.88
2		00 00 00.91	−19 29 55.8	9.27		G	0.003 797 37	−19.498 837 45	+	21.90	181.21	−0.93
3		00 00 01.20	+38 51 33.4	6.61		G	0.005 007 95	+38.859 286 08		2.81	5.24	−2.91
4		00 00 02.01	−51 53 36.8	8.06		H	0.008 381 70	−51.893 546 12		7.75	62.85	0.16
5		00 00 02.39	−40 35 28.4	8.55		H	0.009 965 34	−40.591 224 40		2.87	2.53	9.07

Number HIP (1)	Standard Errors α* mas (14)	δ mas (15)	π mas (16)	$\mu_{\alpha*}$ mas/yr (17)	μ_δ mas/yr (18)	Astrometric Correlations (%) δ/α* (19)	π/α* (20)	π/δ (21)	$\mu_{\alpha*}$/α* (22)	$\mu_{\alpha*}$/δ (23)	$\mu_{\alpha*}$/π (24)	μ_δ/α* (25)	μ_δ/δ (26)	μ_δ/π (27)	μ_δ/$\mu_{\alpha*}$ (28)	Soln F1 % (29)	F2 (30)
1	1.32	0.74	1.39	1.36	0.81	+32	− 7	−11	−24	+ 9	− 1	+10	− 1	+ 1	+34	0	0.74
2	1.28	0.70	3.10	1.74	0.92	+12	−14	−24	−29	+ 1	+21	− 2	−19	−28	+14	2	1.45
3	0.53	0.40	0.63	0.57	0.47	+ 6	+ 9	+ 4	+43	− 1	− 6	+ 3	+24	+ 7	+21	0	−0.45
4	0.53	0.59	0.97	0.65	0.65	−22	− 9	− 3	+24	+20	+ 8	+18	+ 8	−31	−18	0	−1.46
5	0.64	0.61	1.11	0.67	0.74	+10	+24	+ 6	+26	−10	+20	−16	−30	−19	+ 6	0	−1.24

Source parameter covariance matrix: C_{SS}

N astrometric parameters: $\vec{x} = (x_1, \ldots x_N)$

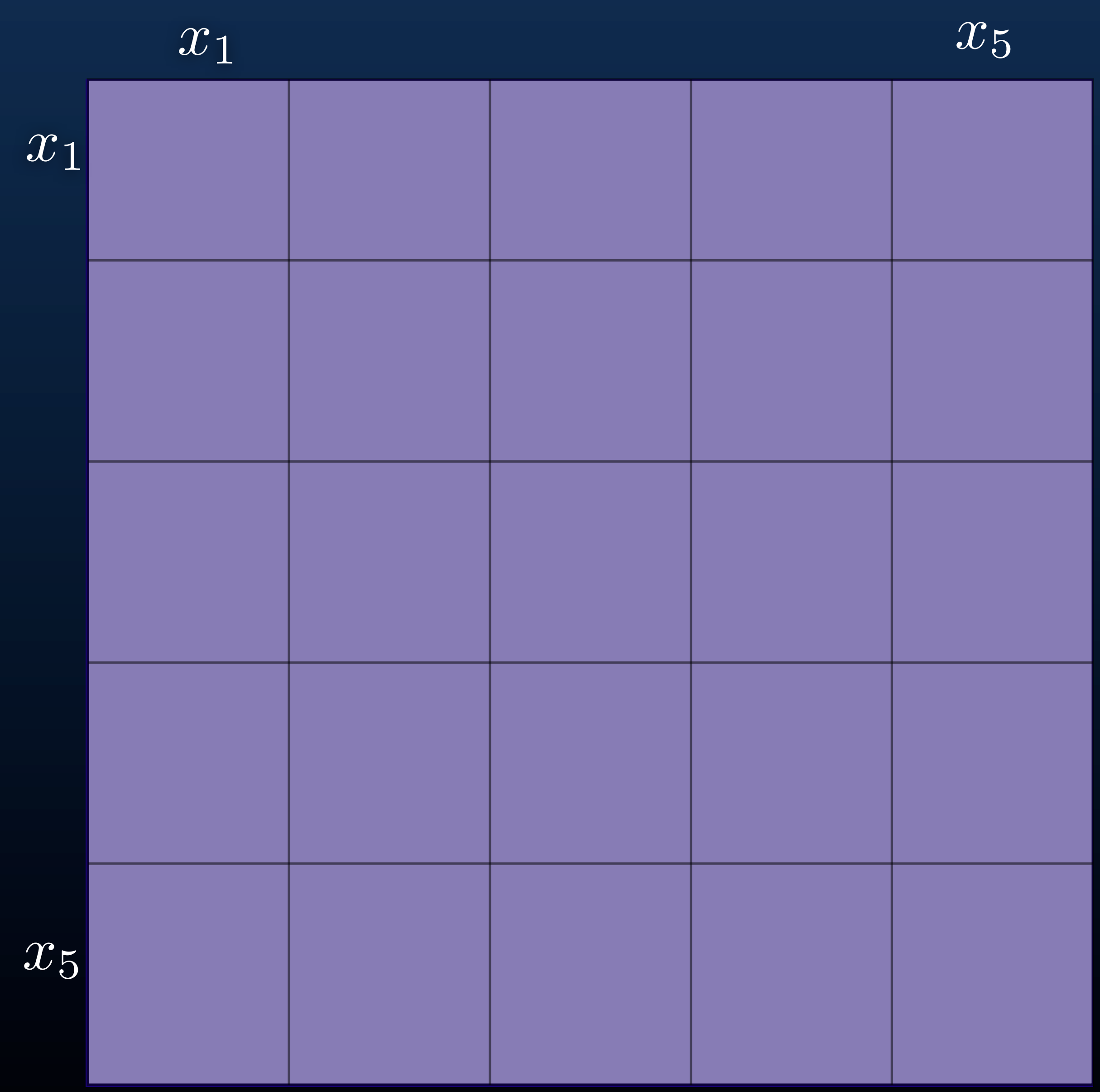
$$\vec{C}_{SS} = \mathrm{Cov}(\vec{x}) =$$



Within-source correlations

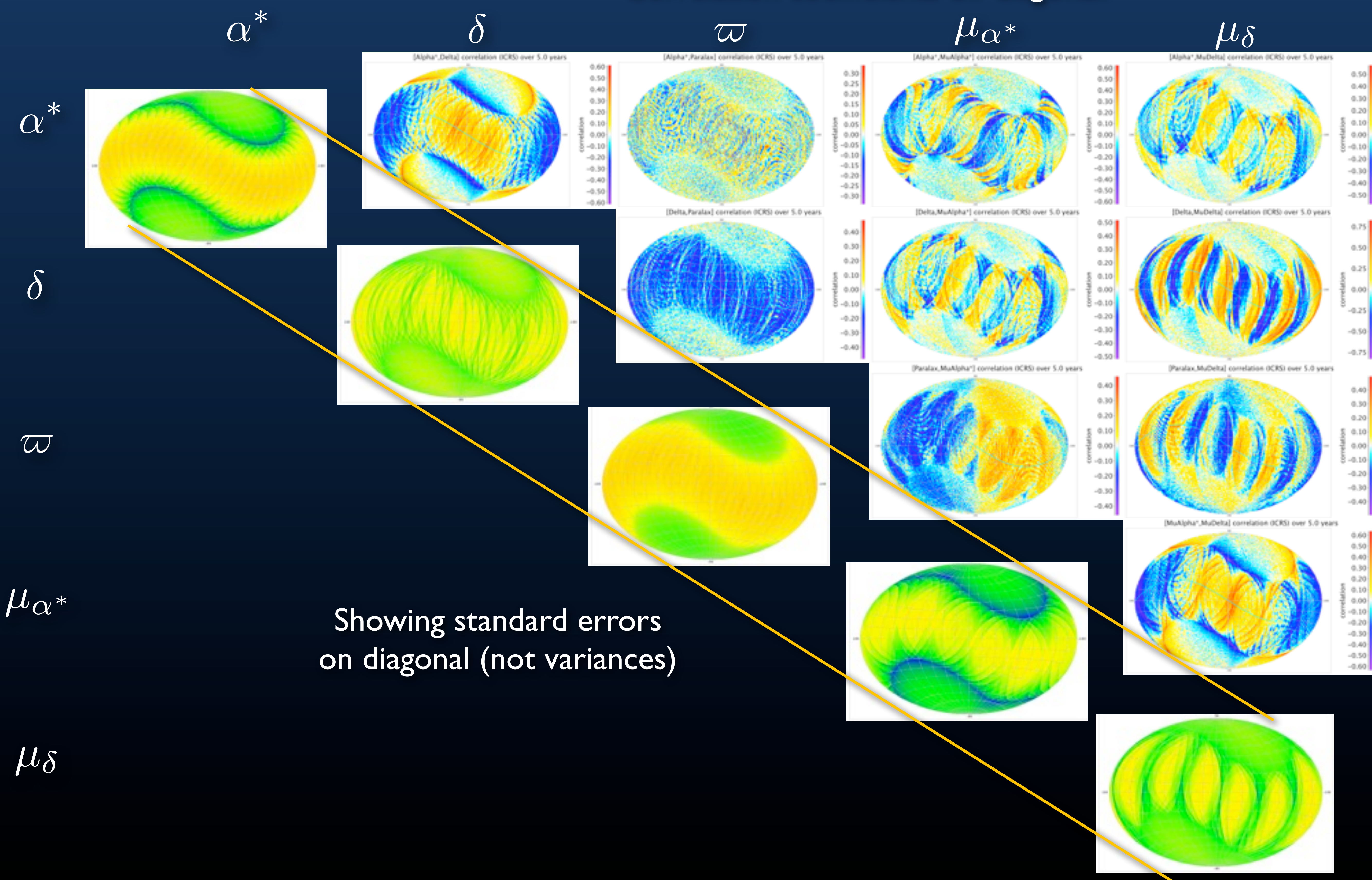
N astrometric parameters: $\vec{x} = (x_1, \ldots x_N)$

Symmetric covariance matrix per source =



Within-source correlations

Correlation coefficients off-diagonal:

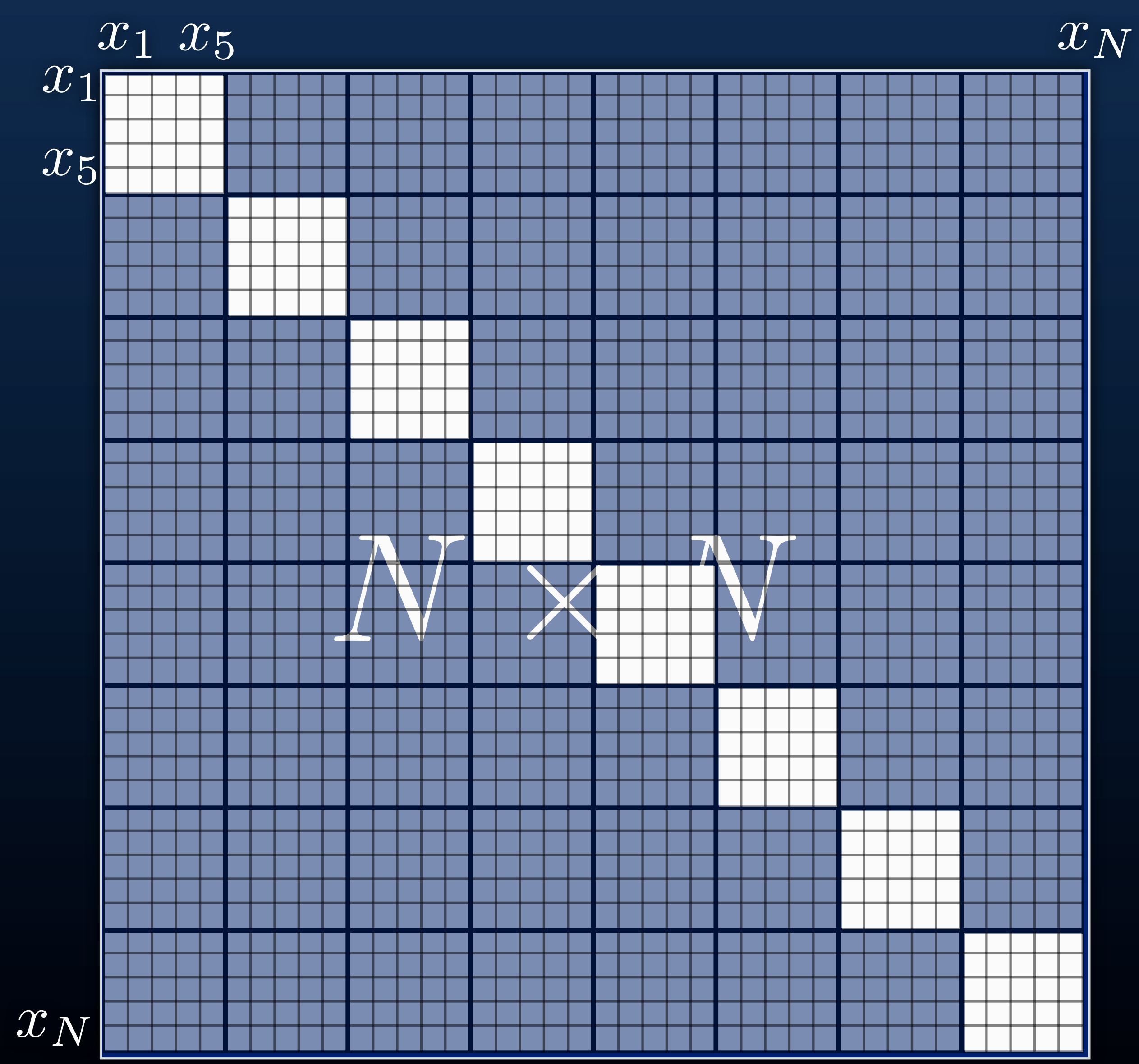


Showing standard errors on diagonal (not variances)

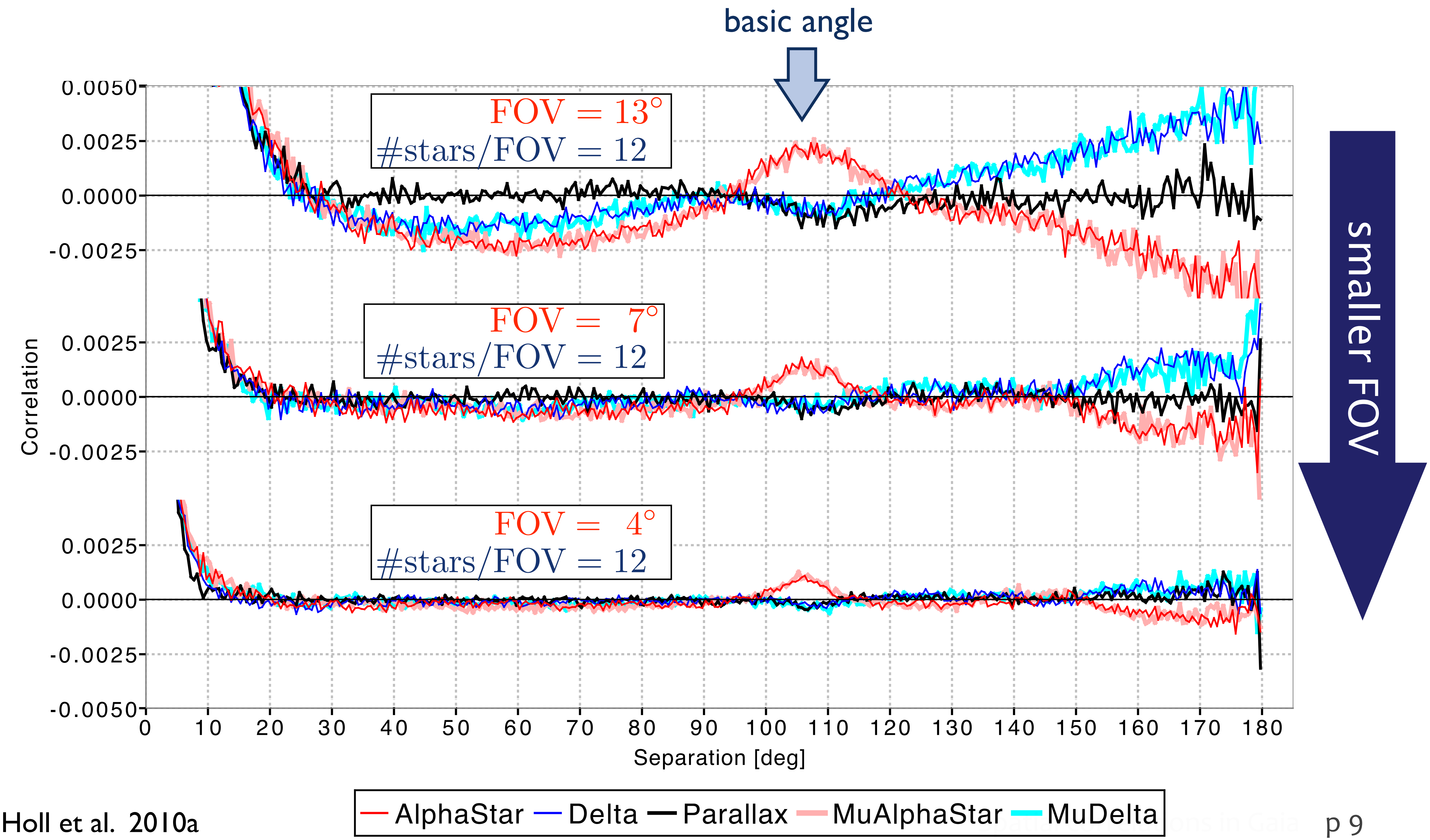
Between-sources correlations

N astrometric parameters: $\vec{x} = (x_1, \ldots x_N)$

$\vec{C}_{SS} = \mathrm{Cov}(\vec{x}) =$

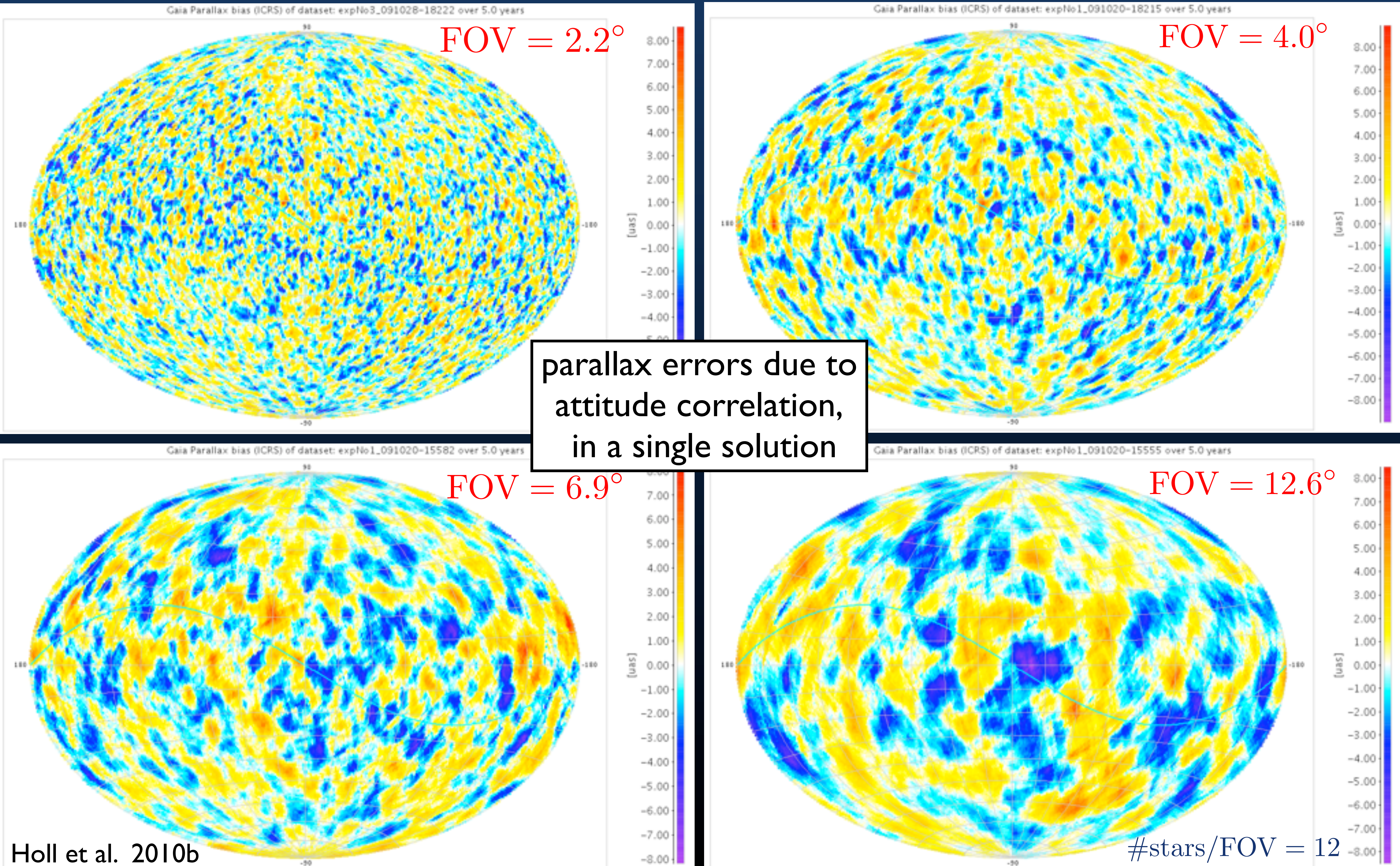


Between-sources correlations (scaled)



Holl et al. 2010a

Spatial correlations due to attitude



parallax errors due to attitude correlation, in a single solution

Holl et al. 2010b

What kind of covariances do we expect between astrometric parameters?

How should we use the covariances?

How can this be done efficiently?

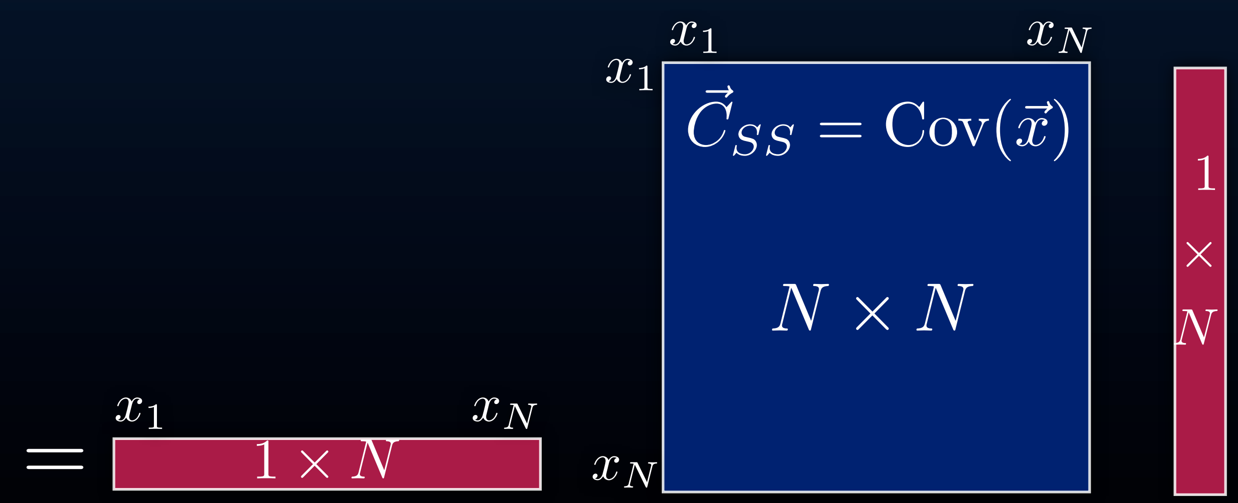
What you want to know

Mean parallax to a star cluster with n stars.

$$\begin{aligned} y &= (\varpi_1 + ... + \varpi_n)/n \\ \sigma_y^2 &= ? \end{aligned}$$

Assuming $y = f(\vec{x})$ is linear for small errors, the variance of y is:

$$\mathrm{Cov}(y) = \sigma_y^2 \quad = \left(\frac{\partial y}{\partial \vec{x}}\right) \vec{C}_{SS} \left(\frac{\partial y}{\partial \vec{x}}\right)' = \sum_i \sum_j \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} (C_{SS})_{ij}$$



What you want to know

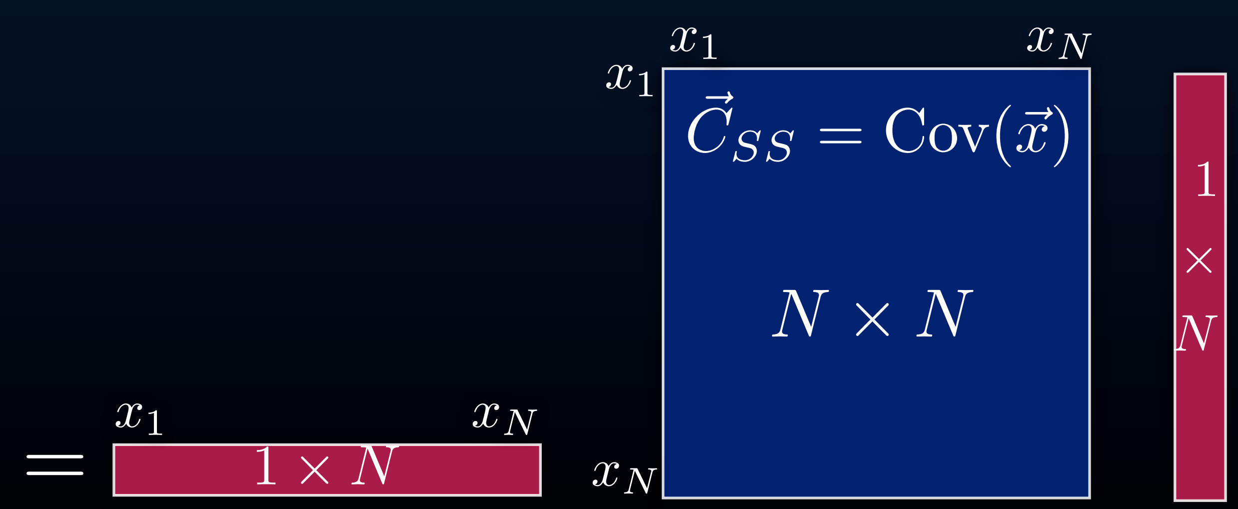
Mean parallax to a star cluster with n stars.

$$y = (\varpi_1 + ... + \varpi_n)/n \qquad\qquad y = f(\vec{x})$$

$$\sigma_y^2 = n^{-2} \sum_{i \in n} \sum_{j \in n} (C_{SS})_{ij}$$

Assuming $y = f(\vec{x})$ is linear for small errors, the variance of y is:

$$\mathrm{Cov}(y) = \sigma_y^2 \quad = \left(\frac{\partial y}{\partial \vec{x}}\right) \vec{C}_{SS} \left(\frac{\partial y}{\partial \vec{x}}\right)' = \sum_i \sum_j \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} (C_{SS})_{ij}$$

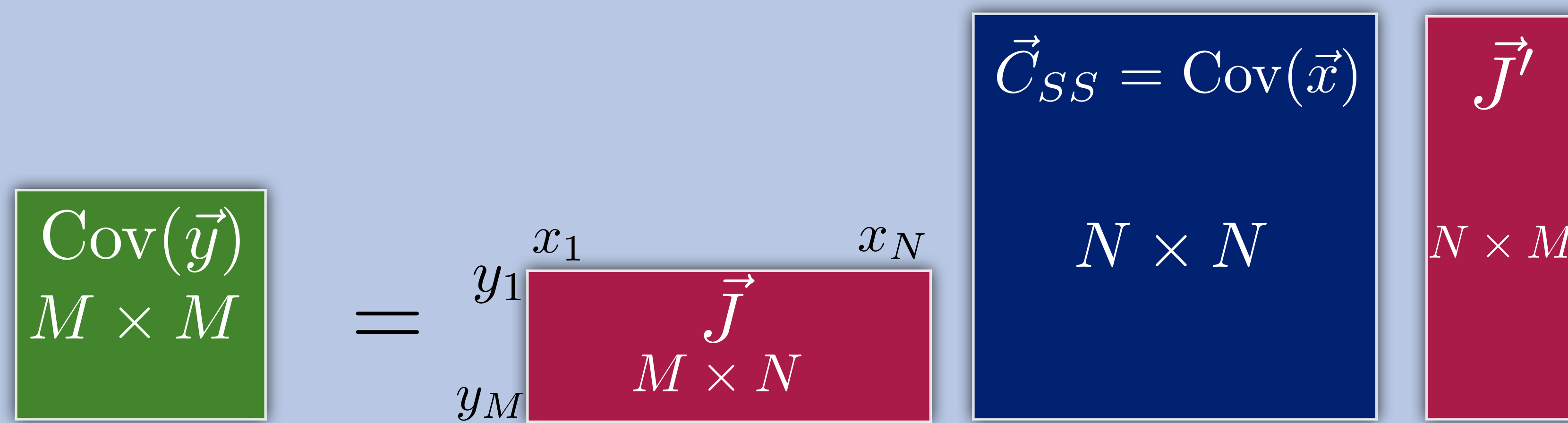


What you want to know

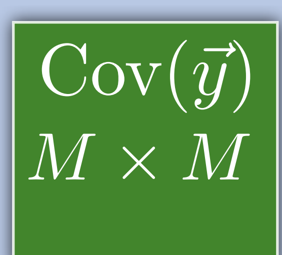
In general for M different scalar quantities:

$$\vec{y} = (y_1, \ldots, y_M) \qquad y_m = f_m(\vec{x})$$

$$\mathrm{Cov}(\vec{y}) = \vec{J}\vec{C}_{SS}\vec{J}' \qquad J_{mj} = \partial y_m / \partial x_j$$

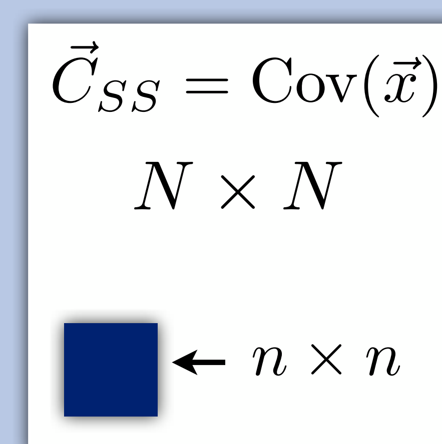


Only consider n active parameters (non-zero columns in $\vec{J}$), then typically:



$$M(M-1)/2 \quad \ll \quad n(n-1)/2$$

(number of non-redundant elements)



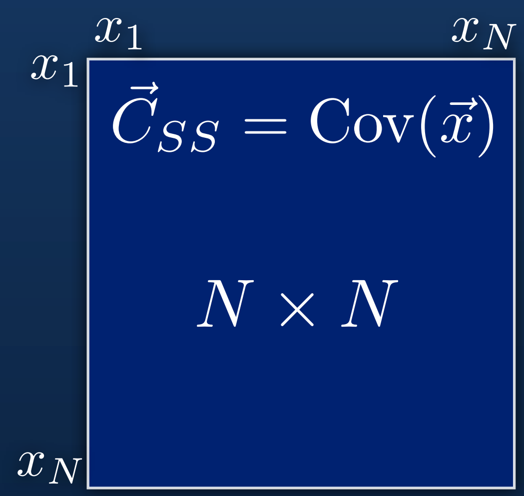
What kind of covariances do we expect between astrometric parameters?

How should we use the covariances?

How can this be done efficiently?

How large is C_{SS}?

- $N = 5 \times 10^9$ for 10^9 sources (so ~10^{19} elements).
- Takes ~ **10^8 TeraByte** of space!
- Computing directly from normal matrix is unfeasible (Bombrun et al. 2011).



Its **clearly** desirable that the covariance, between any pair of astrometric parameters, can be **computed** from a **reduced** amount of data.

How to compute $(\vec{C}_{SS})_{ij}$ **accurately** and **efficiently**?

How do we actually estimate astrometric parameters?

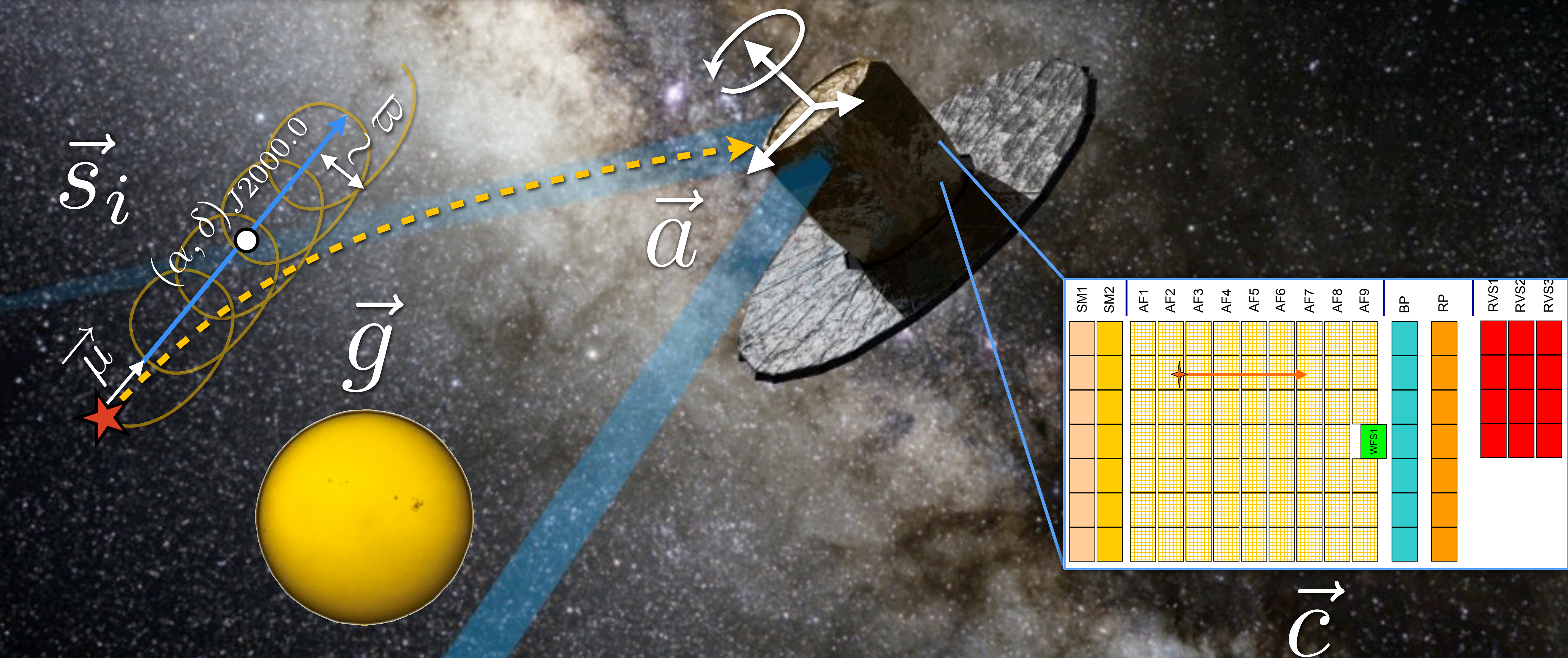
Formula alert!
(stay right for easy passing...)

How to compute $(\vec{C}_{SS})_{ij}$ **accurately** and **efficiently**?

Astrometric parameters

'Nuisance' parameters (from astrometric point of view):

- $\vec{a}$ attitude
- $\vec{c}$ calibration
- $\vec{g}$ global



Gaia model & image rendering by Berry Holl

How to estimate astrometric parameters?

Least squares solution with ~10^9 parameters!

$$\min_{\vec{s},\vec{a},\vec{c},\vec{g}} \sum_l \left[\frac{t_l - f_l(\vec{s}_i, \vec{a}, \vec{c}, \vec{g})}{\sigma_l} \right]^2$$

$t_l =$ measured observation time l (with uncertainty σ_l)
$f_l =$ modelled observation time l

In this talk **we ignore** $\vec{c}$ **and** $\vec{g}$ since they are affected by large number of observations over large part of the sphere.

A direct solution is unfeasible (Bombrun et al. 2011), therefore solved iteratively using **AGIS** (Lindegren et al. 2011): **A**strometric **G**lobal **I**terative **S**olution

Normal matrix

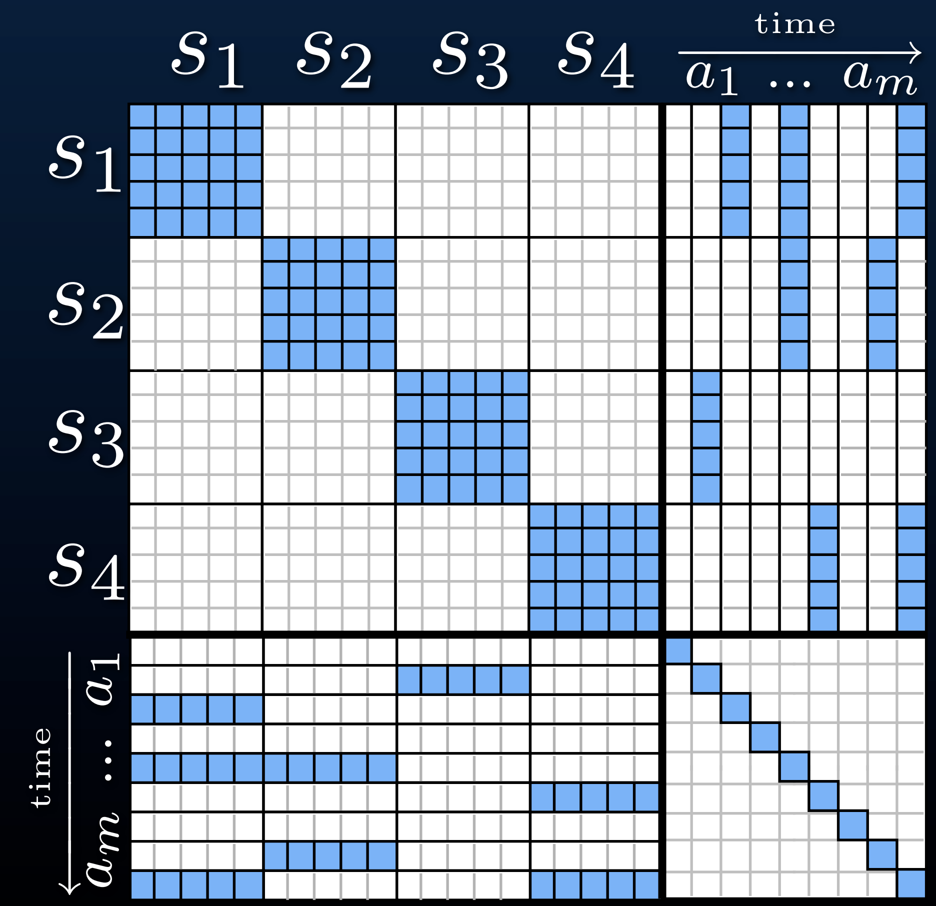
Least-squares estimation of $\vec{s}$ and $\vec{a}$

$$\forall l : \quad \vec{S}_l \; \Delta\vec{s}_i + \vec{A}_l \; \Delta\vec{a} = h_l$$

$$\vec{S}_l = (\partial f_l / \partial \vec{s}_i')/\sigma_l$$
$$\vec{A}_l = (\partial f_l / \partial \vec{a}')/\sigma_l$$
$$h_l = (t_l - f_l(\vec{s}_i, \vec{a}))\,/\sigma_l$$

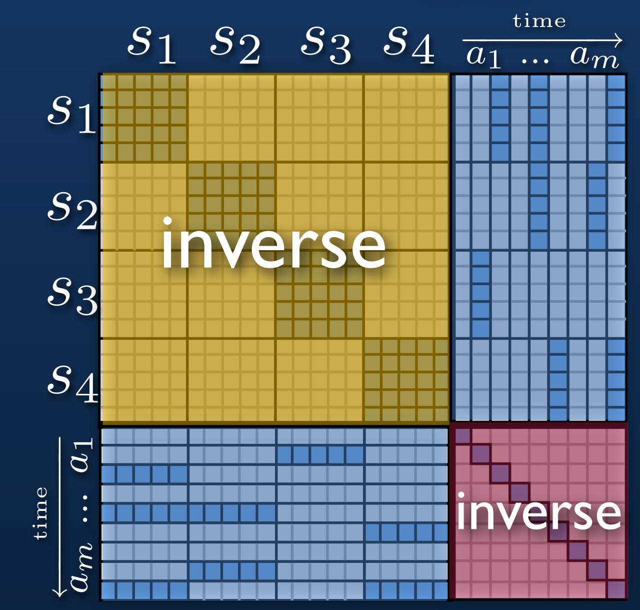
observation equations:
for each observations,
how to modify parameters
to get zero residual between
measured and modelled time.

$$\vec{N} = \begin{bmatrix} \vec{N}_{SS} & \vec{N}_{SA} \\ \vec{N}_{AS} & \vec{N}_{AA} \end{bmatrix} =$$



Source parameter covariance C_{SS}

normal $\vec{N} = \begin{bmatrix} \vec{N}_{SS} & \vec{N}_{SA} \\ \vec{N}_{AS} & \vec{N}_{AA} \end{bmatrix} =$



covariance $\vec{C} = \begin{bmatrix} \vec{C}_{SS} & \vec{C}_{SA} \\ \vec{C}_{AS} & \vec{C}_{AA} \end{bmatrix} = \vec{N}^{-1}$

unfeasible!

Expansion of source parameter covariance*:

$$\vec{C}_{SS} = \vec{N}_{SS}^{-1} + \vec{N}_{SS}^{-1}\vec{N}_{SA}\left(\vec{N}_{AA}^{-1}\right)\vec{N}_{AS}\vec{N}_{SS}^{-1} + \vec{N}_{SS}^{-1}\vec{N}_{SA}\left(\vec{N}_{AA}^{-1}\vec{N}_{AS}\left(\vec{N}_{SS}^{-1}\right)\vec{N}_{SA}\vec{N}_{AA}^{-1}\right)\vec{N}_{AS}\vec{N}_{SS}^{-1} + \ldots$$

$$= \vec{C}_{SS}^{(1)} + \vec{C}_{SS}^{(2)} + \vec{C}_{SS}^{(3)} + \ldots + \vec{C}_{SS}^{(p)} + \ldots$$

**Holl et al. 2011, in prep.*

Source parameter covariance C_{SS}

$(\vec{C}^{(1)}_{SS})_{ij}$ = Only estimating source parameters from observations assuming attitude is known.
No source coupling, so zero for $i \neq j$.

$(\vec{C}^{(2)}_{SS})_{ij}$ = Only estimating attitude parameters from observations and propagating those attitude cov back to source cov.
Sources get coupled to attitude parameters at which they were observed.

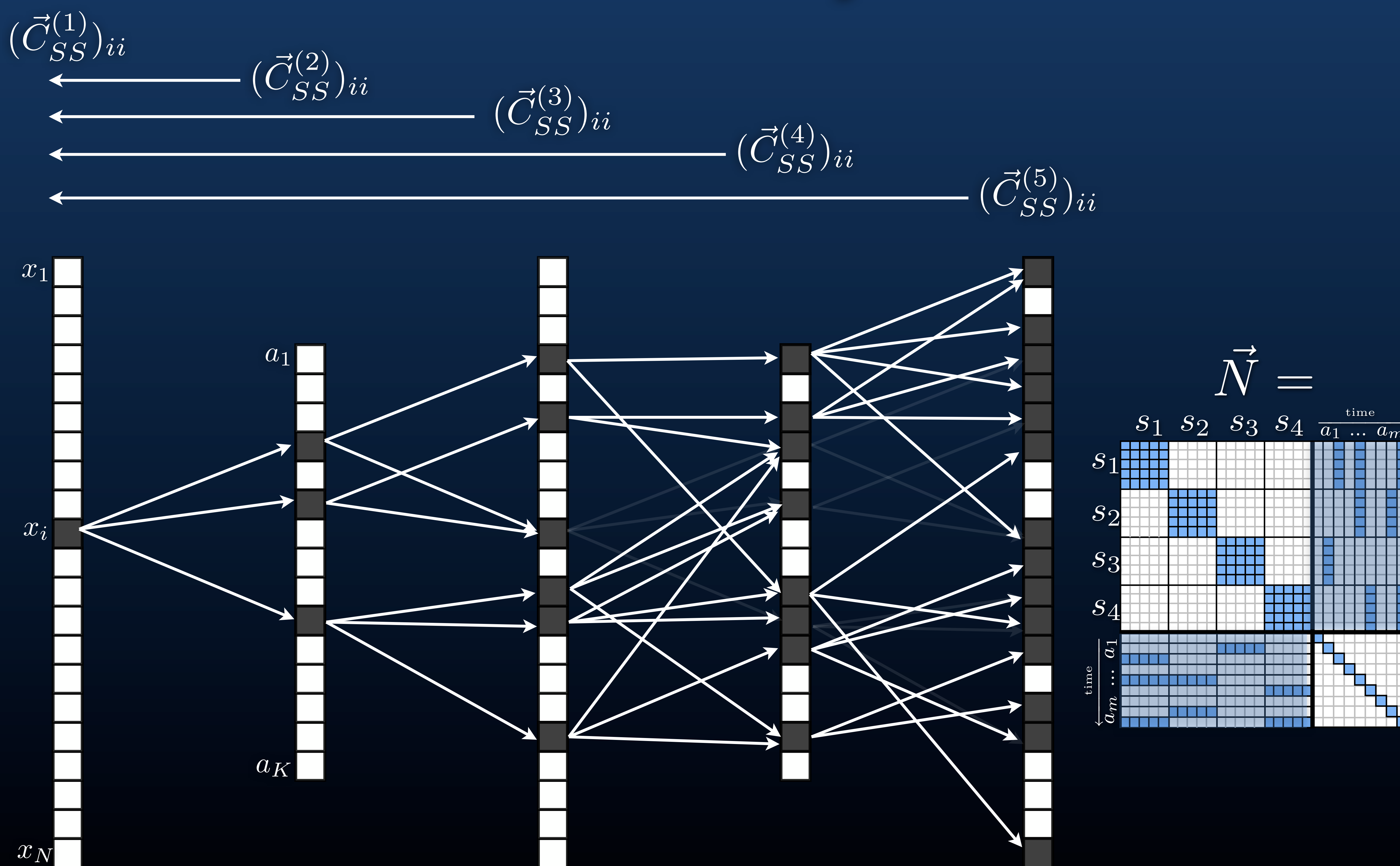
Expansion of source parameter covariance*:

$$\vec{C}_{SS} = \vec{N}^{-1}_{SS} + \vec{N}^{-1}_{SS}\vec{N}_{SA}\left(\vec{N}^{-1}_{AA}\right)\vec{N}_{AS}\vec{N}^{-1}_{SS} + \vec{N}^{-1}_{SS}\vec{N}_{SA}\left(\vec{N}^{-1}_{AA}\vec{N}_{AS}\left(\vec{N}^{-1}_{SS}\right)\vec{N}_{SA}\vec{N}^{-1}_{AA}\right)\vec{N}_{AS}\vec{N}^{-1}_{SS} + \ldots$$

$$= \vec{C}^{(1)}_{SS} + \vec{C}^{(2)}_{SS} + \vec{C}^{(3)}_{SS} + \ldots + \vec{C}^{(p)}_{SS} + \ldots$$

*Holl et al. 2011, in prep.

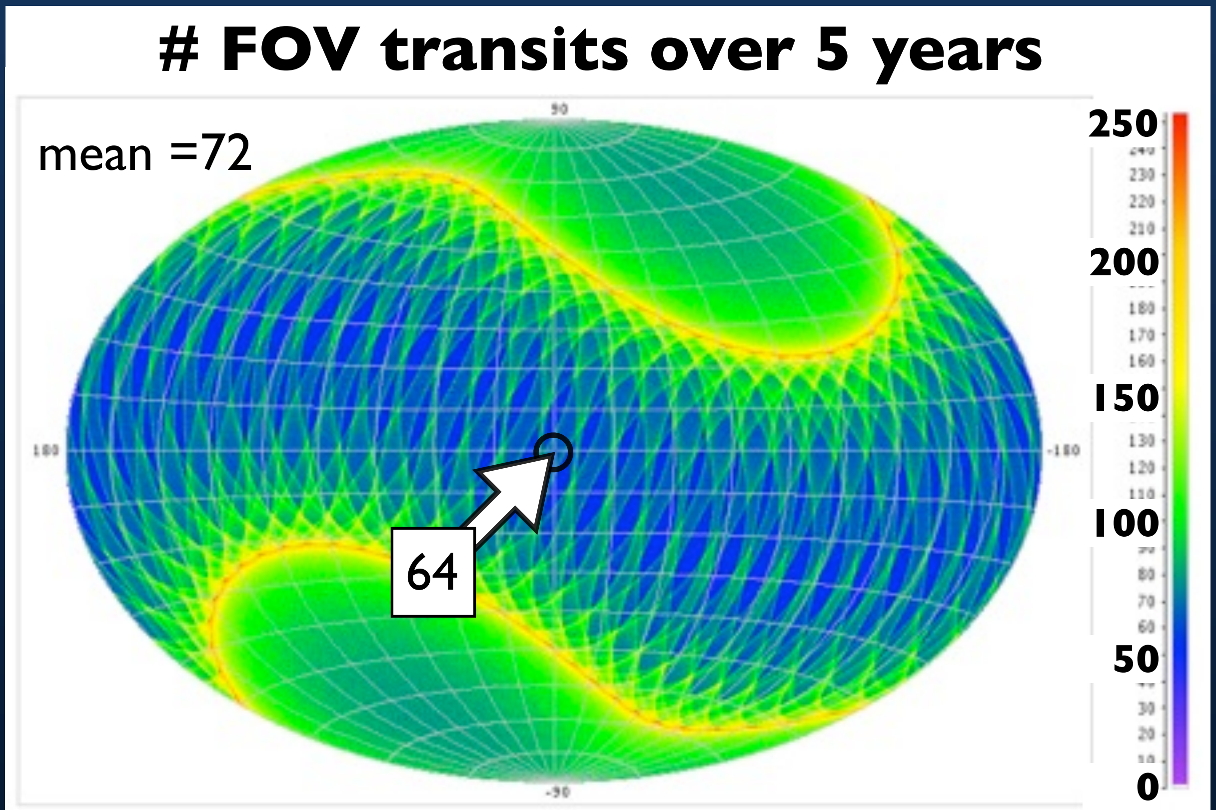
Parameters connectivity



Parameters connectivity study

All connections for each term p:

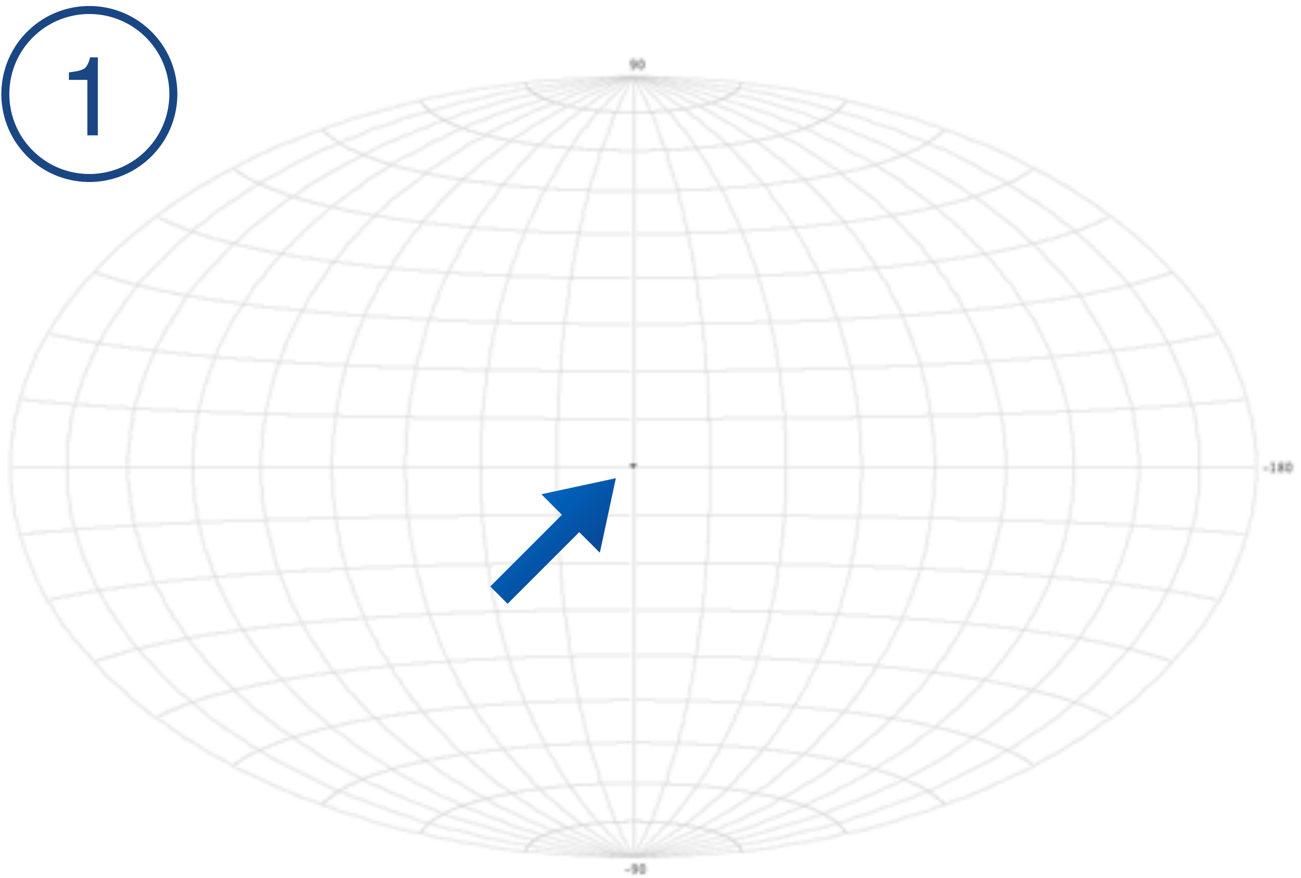
- $(0°, 0°)$ ecliptic, 64 FOV transits.
- 196,608 sources, ~0.46° separation (Gaia FOV ~0.7x0.7°).
- 2,629,800 attitude intervals, 60 sec per interval (Gaia FOV transit time ~45 sec).
- Simulated 5 years observations.



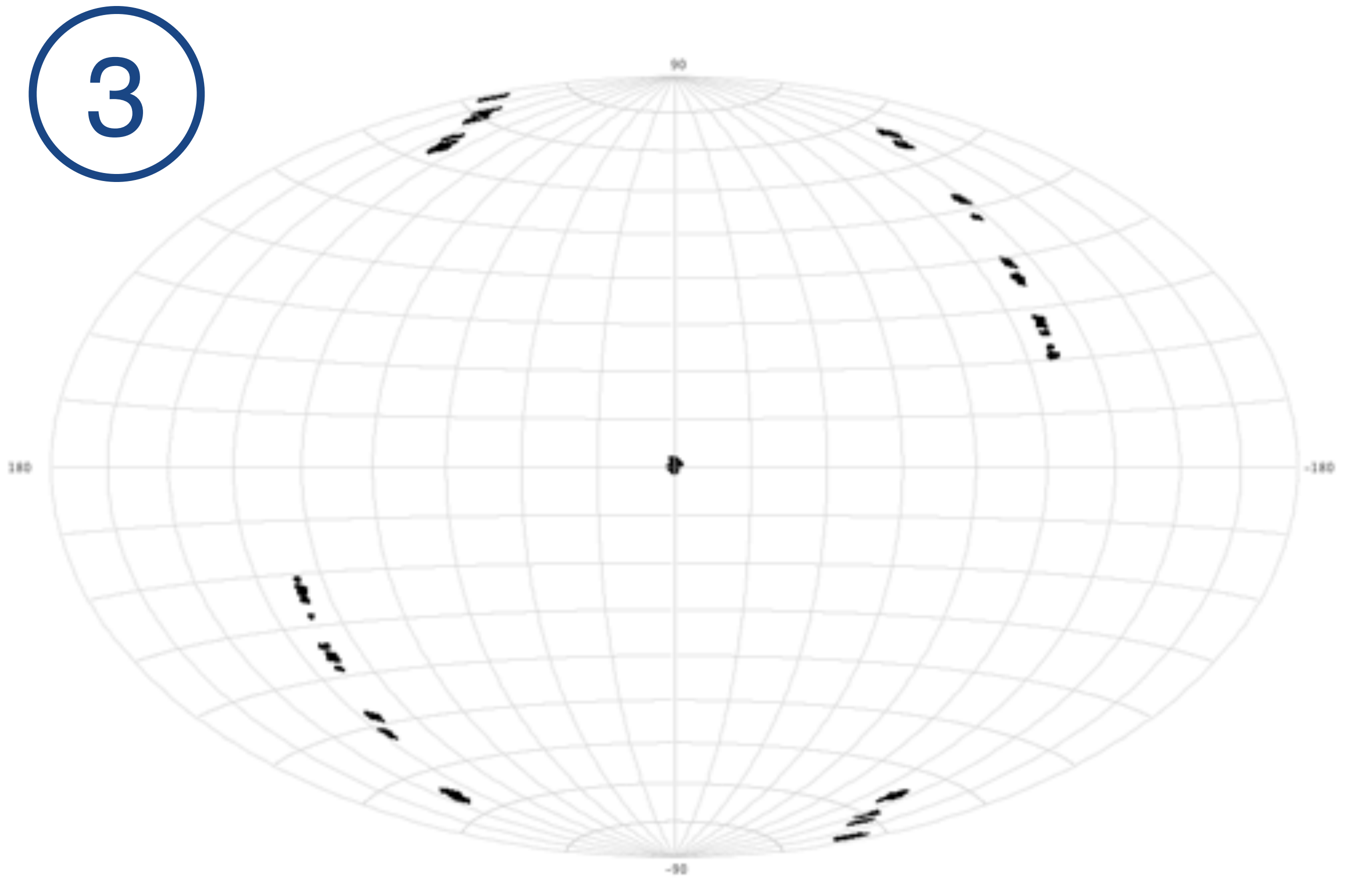
AGISLab
Investigating the Astrometric Global Iterative Solution

Source connections to (0°, 0°)

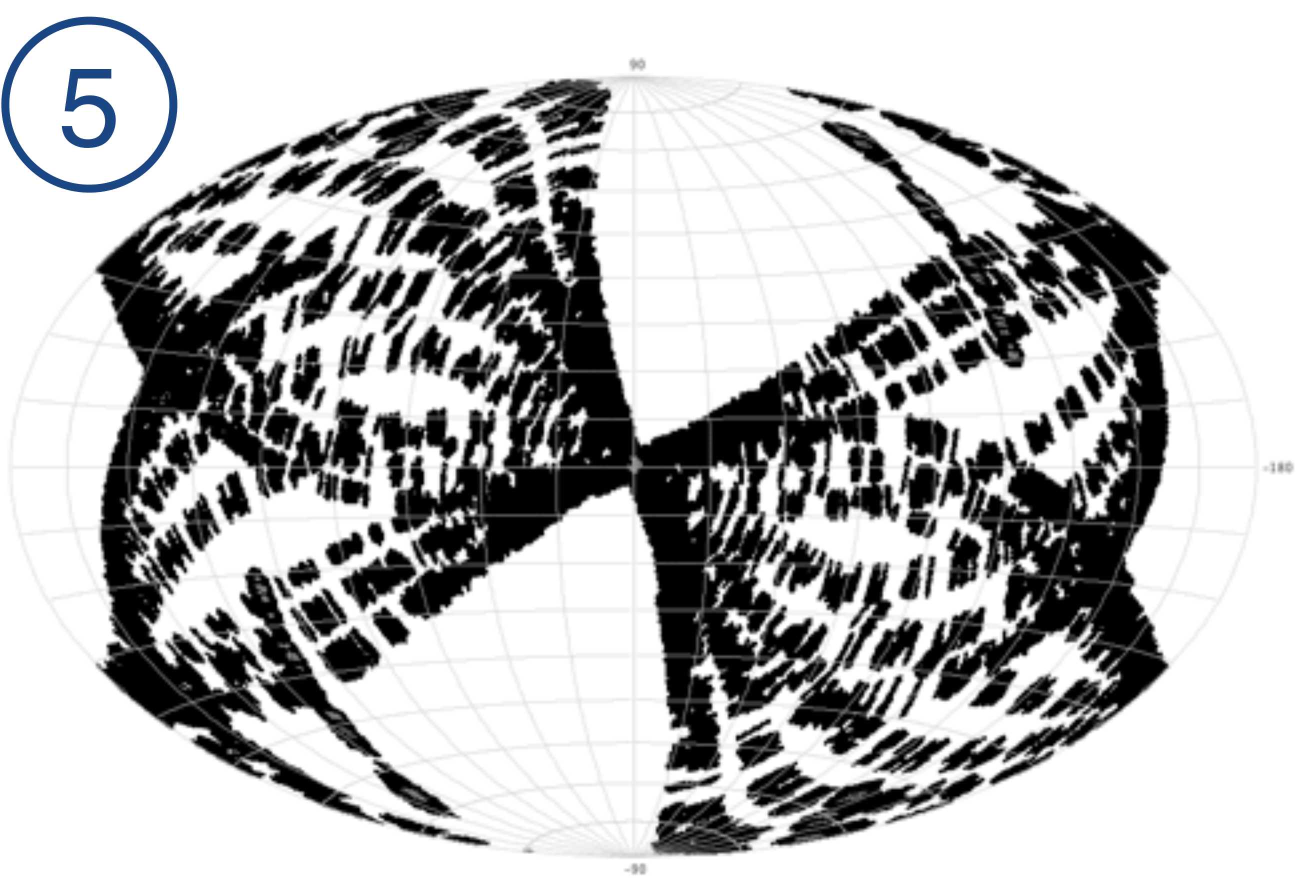
Connections Cov term 1 alpha: 0.0 delta: 0.0 (0.00% of total 196608 sources)



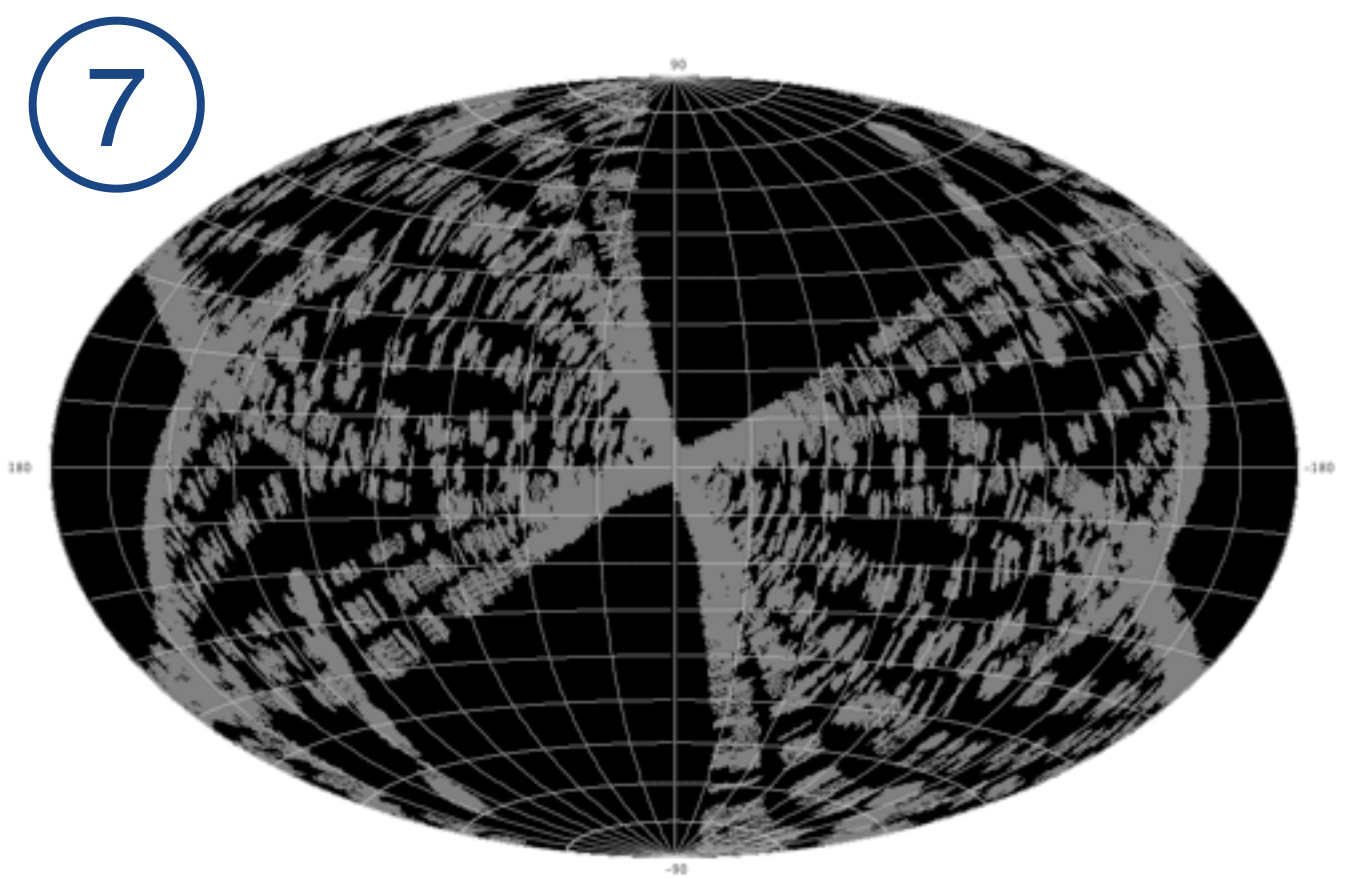
Connections Cov term 3 alpha: 0.0 delta: 0.0 (0.24% of total 196608 sources)



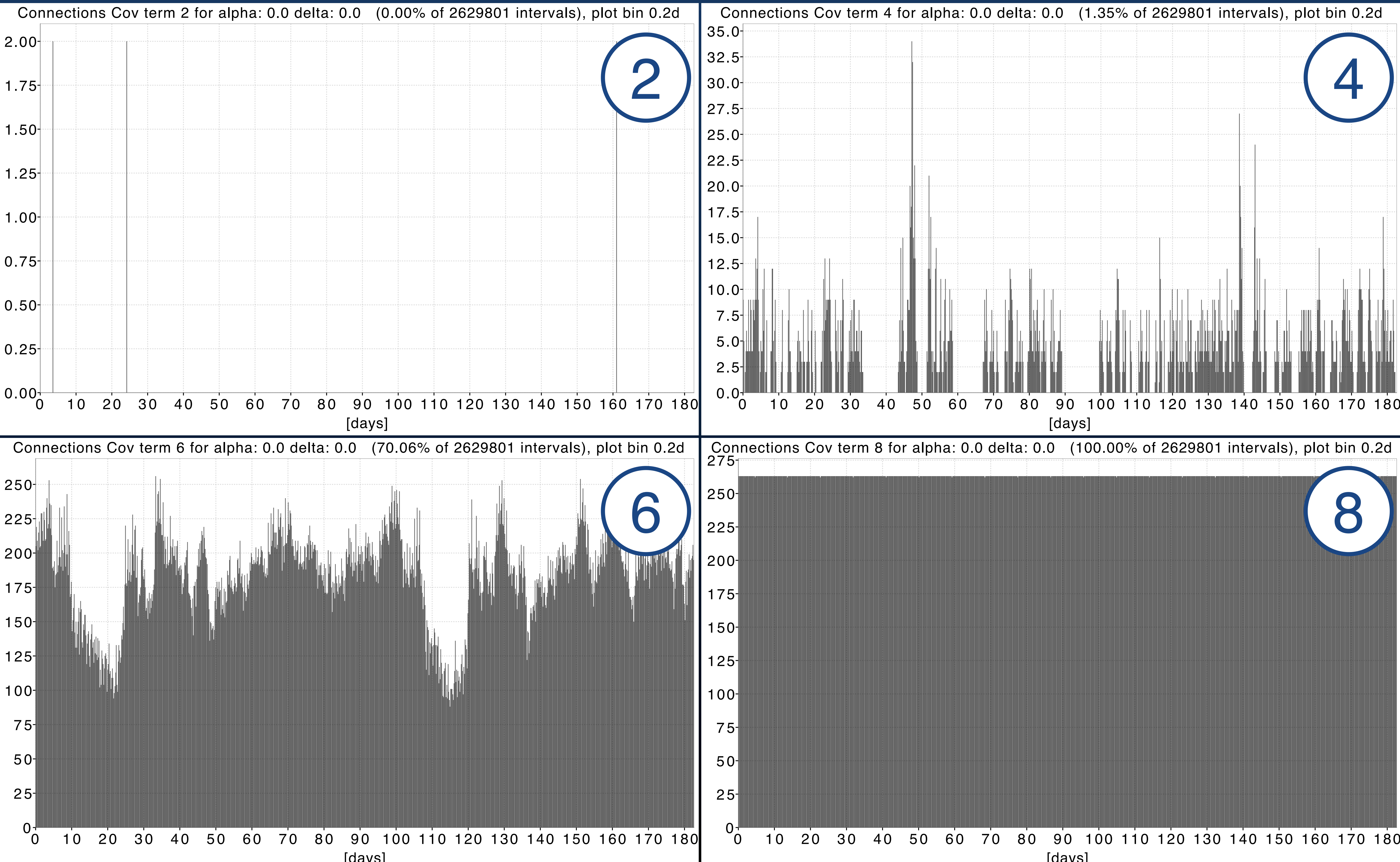
Connections Cov term 5 alpha: 0.0 delta: 0.0 (32.56% of total 196608 sources)



Connections Cov term 7 alpha: 0.0 delta: 0.0 (100.00% of total 196608 sources)



Attitude connections to (0°, 0°)



plot bin size 0.18 days (containing 263 attitude intervals)

Covariance expansion model

Input data:

- Per source, per field-of-view (FOV) transit:
  - partial derivatives of observations w.r.t. the (5) source parameters.
  - observation time
  - combined weight of the observations

On average 72 FOV transits ~ 2 TB for 10^9 sources.

~10^8 times less than storing the full covariance matrix!

Conclusions

- No chance to provide the whole $\vec{C}_{SS}$.
- We found recursive algorithm to compute any element of $\vec{C}_{SS}$ (based on reduced amount of structural data), to any accuracy at increasing computational cost.
- But in reality we want $\mathrm{Cov}(\vec{y})$, which might involve a very large number of elements of $\vec{C}_{SS}$, making it impractical/unfeasible, so:
- If the user provides $\vec{J}$, apply recursive algorithm to compute $\mathrm{Cov}(\vec{y}) = \vec{J}\vec{C}_{SS}\vec{J}'$ directly, without explicitly computing any part of $\vec{C}_{SS}$ (is possible).

Contact details

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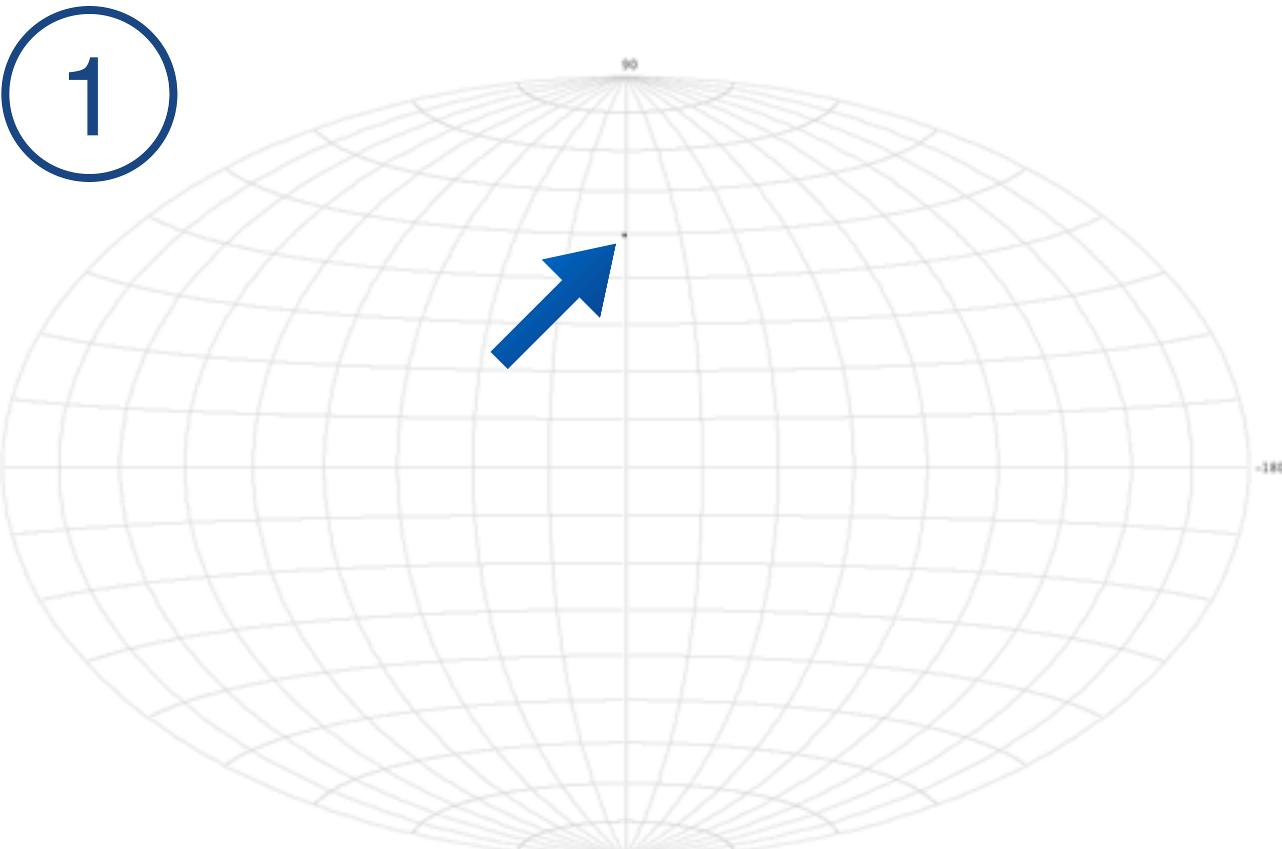
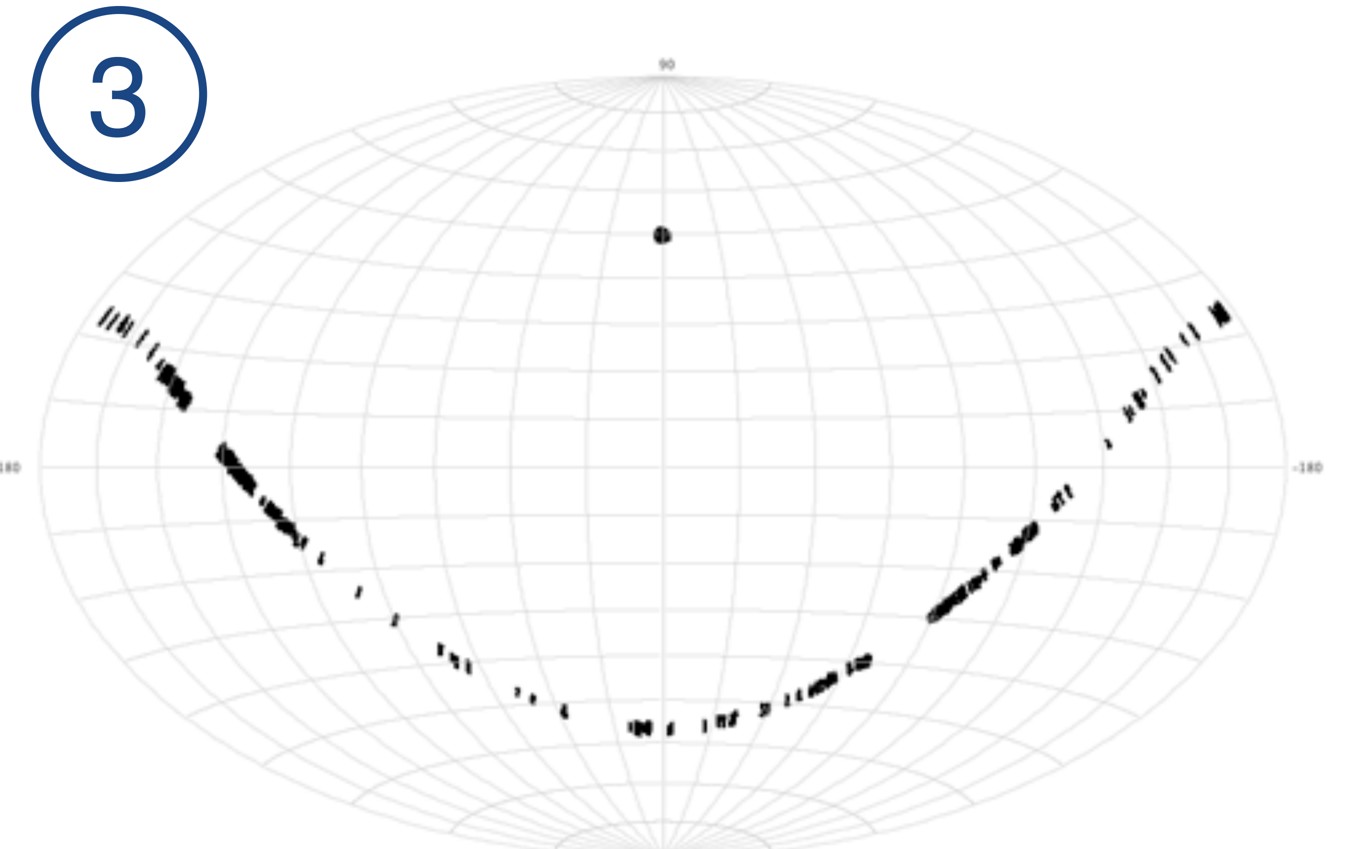
Appendices

Source connections to (0°, 50°)

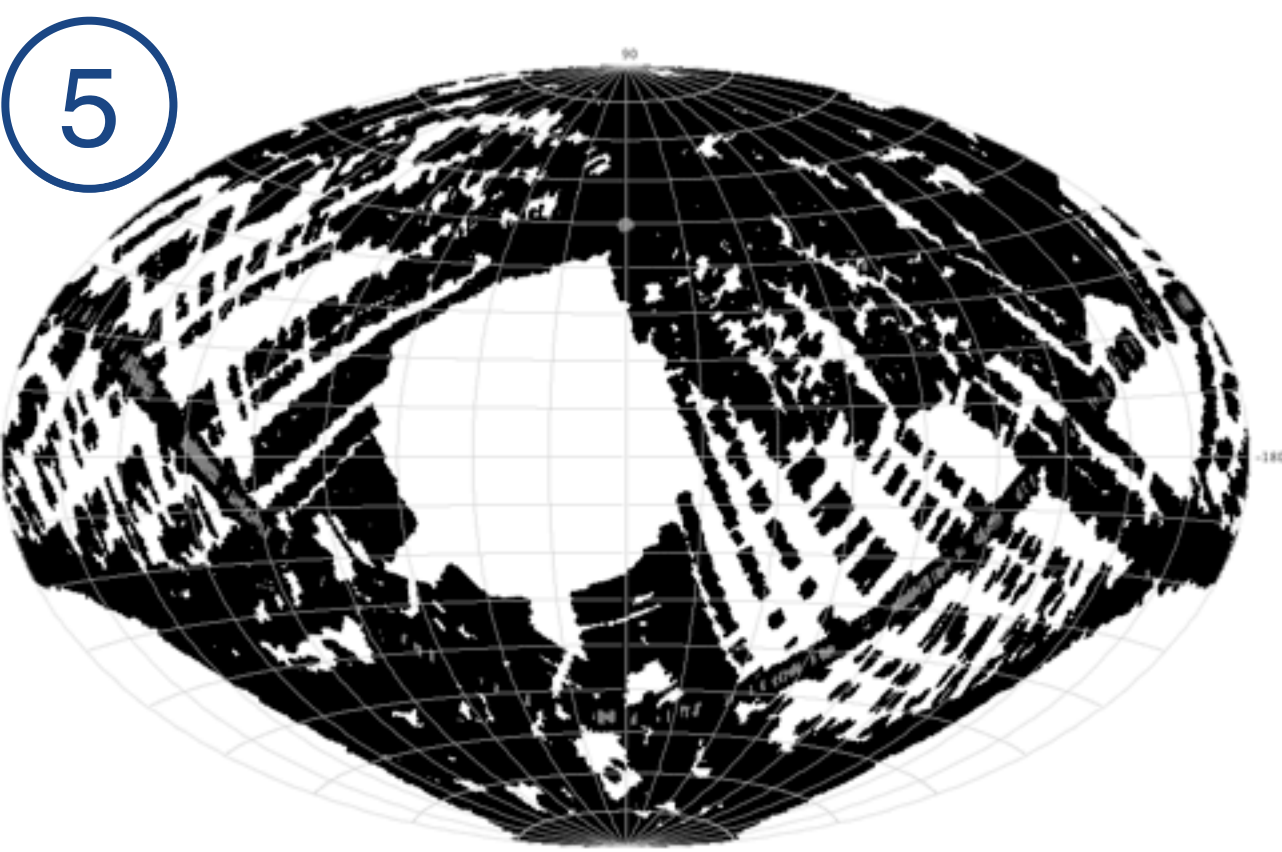
Connections Cov term 1   alpha: 0.0 delta: 50.0   (0.00% of total 196608 sources)



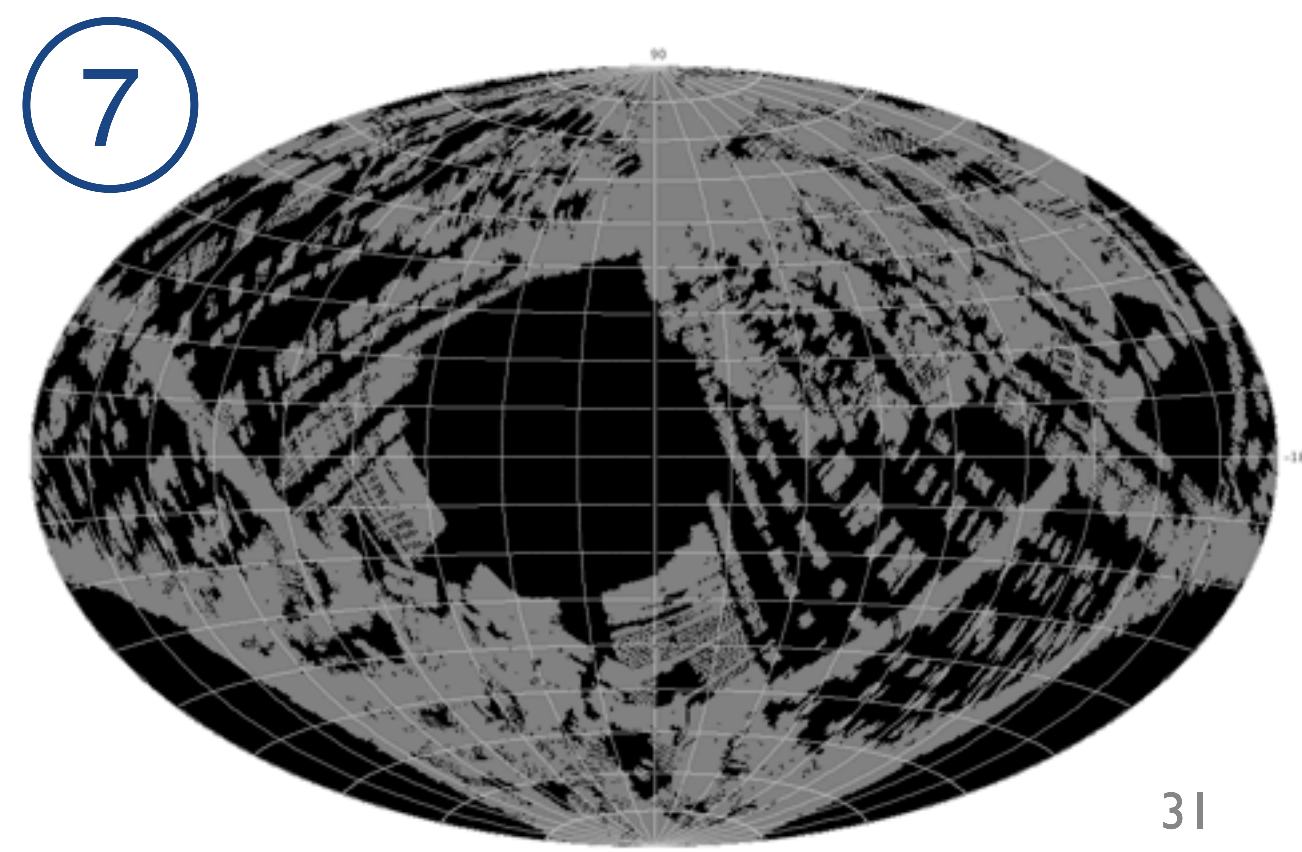
Connections Cov term 3   alpha: 0.0 delta: 50.0   (0.66% of total 196608 sources)



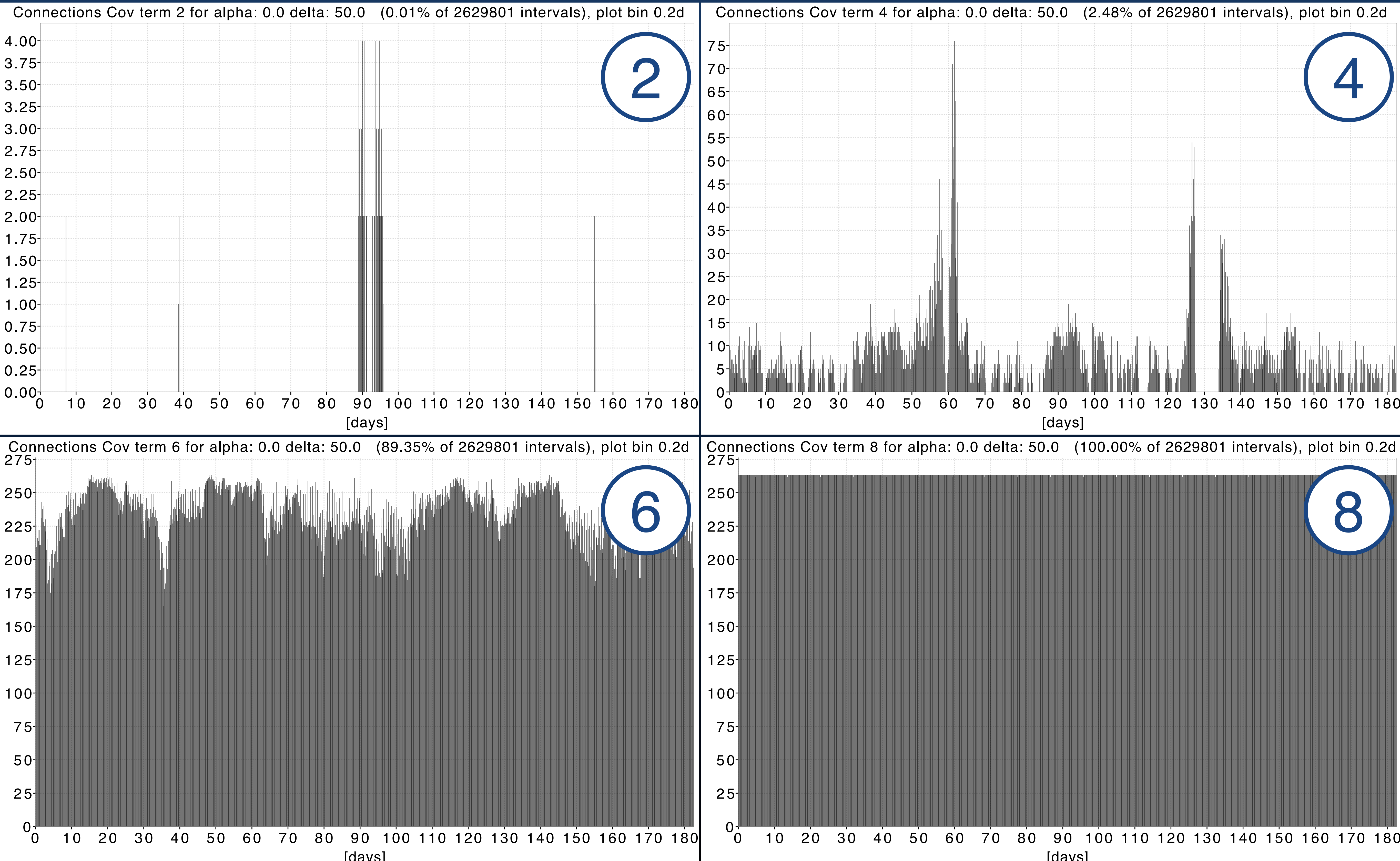
Connections Cov term 5   alpha: 0.0 delta: 50.0   (49.50% of total 196608 sources)



Connections Cov term 7   alpha: 0.0 delta: 50.0   (100.00% of total 196608 sources)



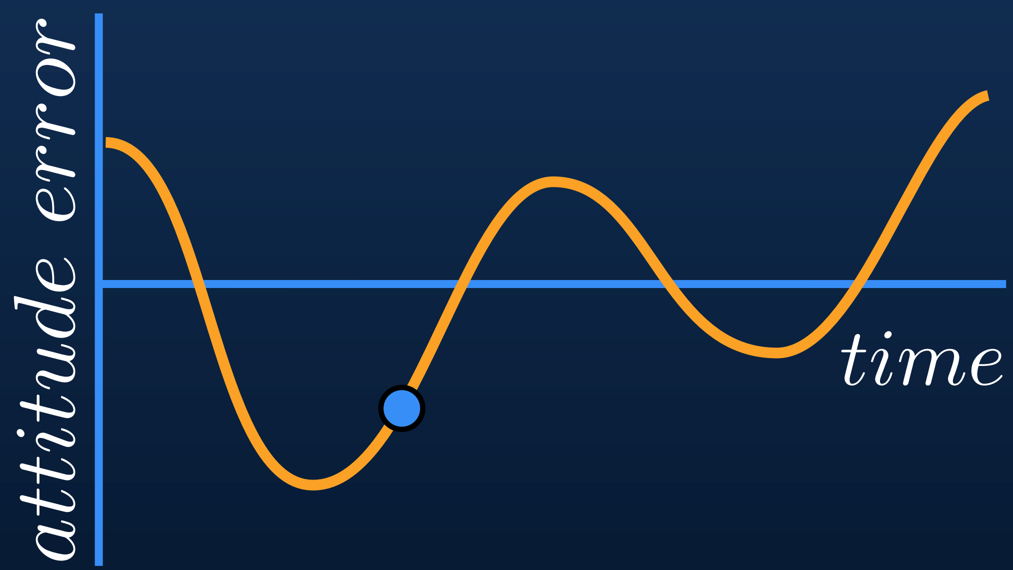
Attitude connections to (0°, 50°)



plot bin size 0.18 days (containing 263 attitude intervals)

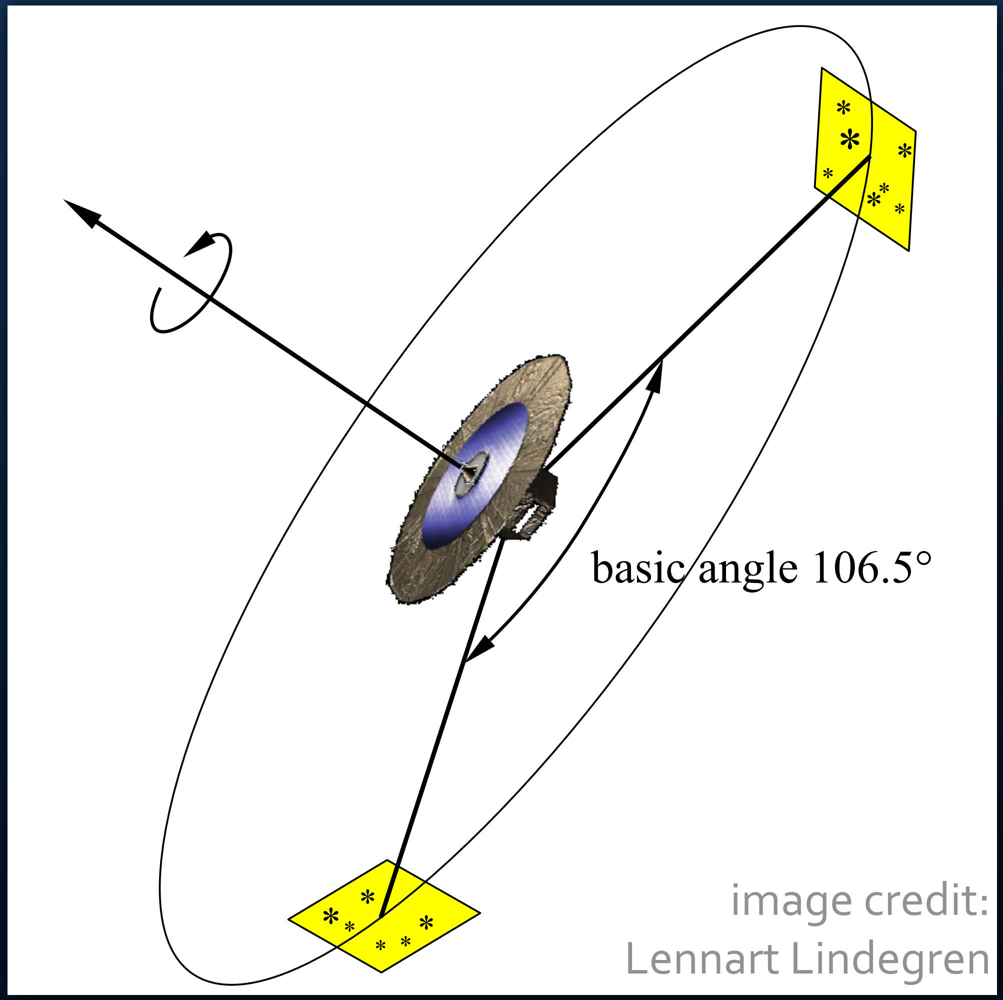
Theory: attitude causes spatial correlations

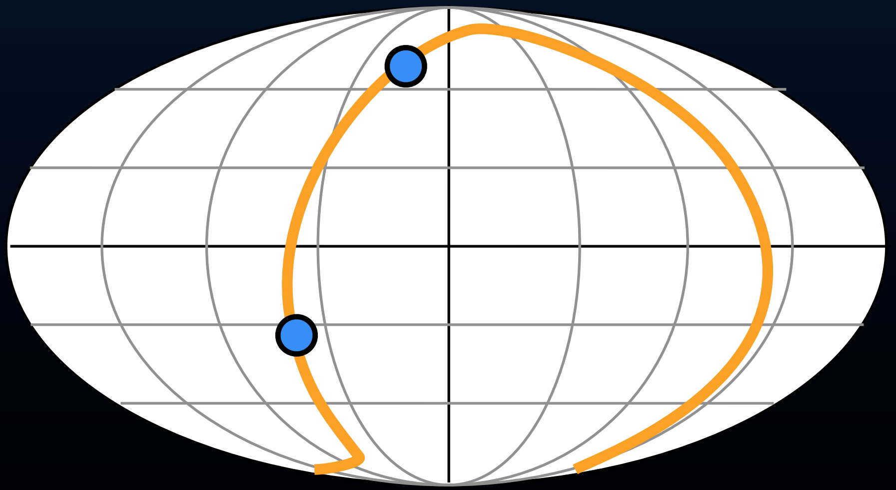
Attitude is solved as part of solution, so it cannot be **exactly** determined!



For **attitude error at time t**, all observations in the focal plane will be affected:

- Correlation between stars **in the FOV**.
- Correlation between stars **in the fields separated by the basic angle**.





Gaia scanning law

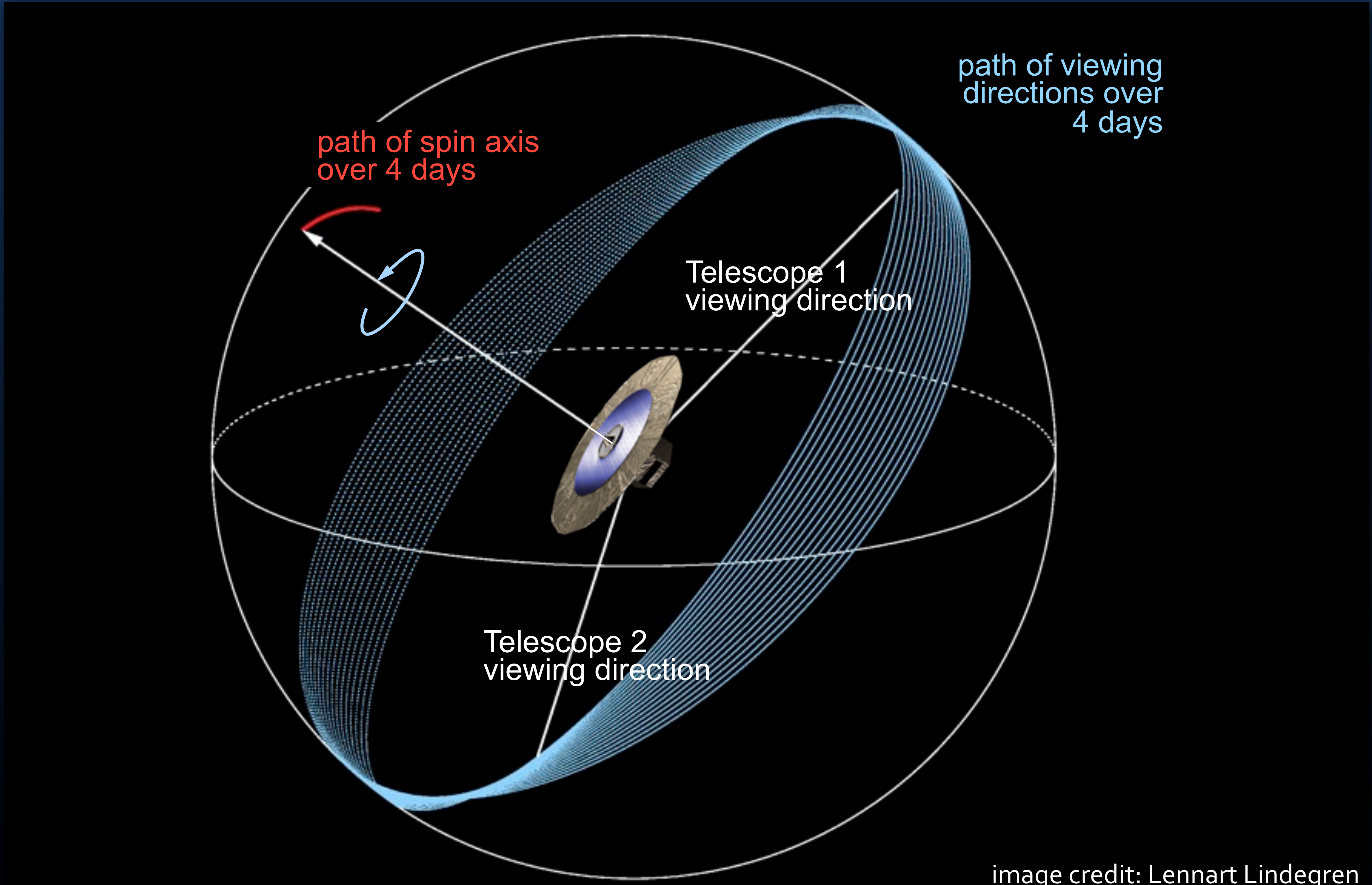
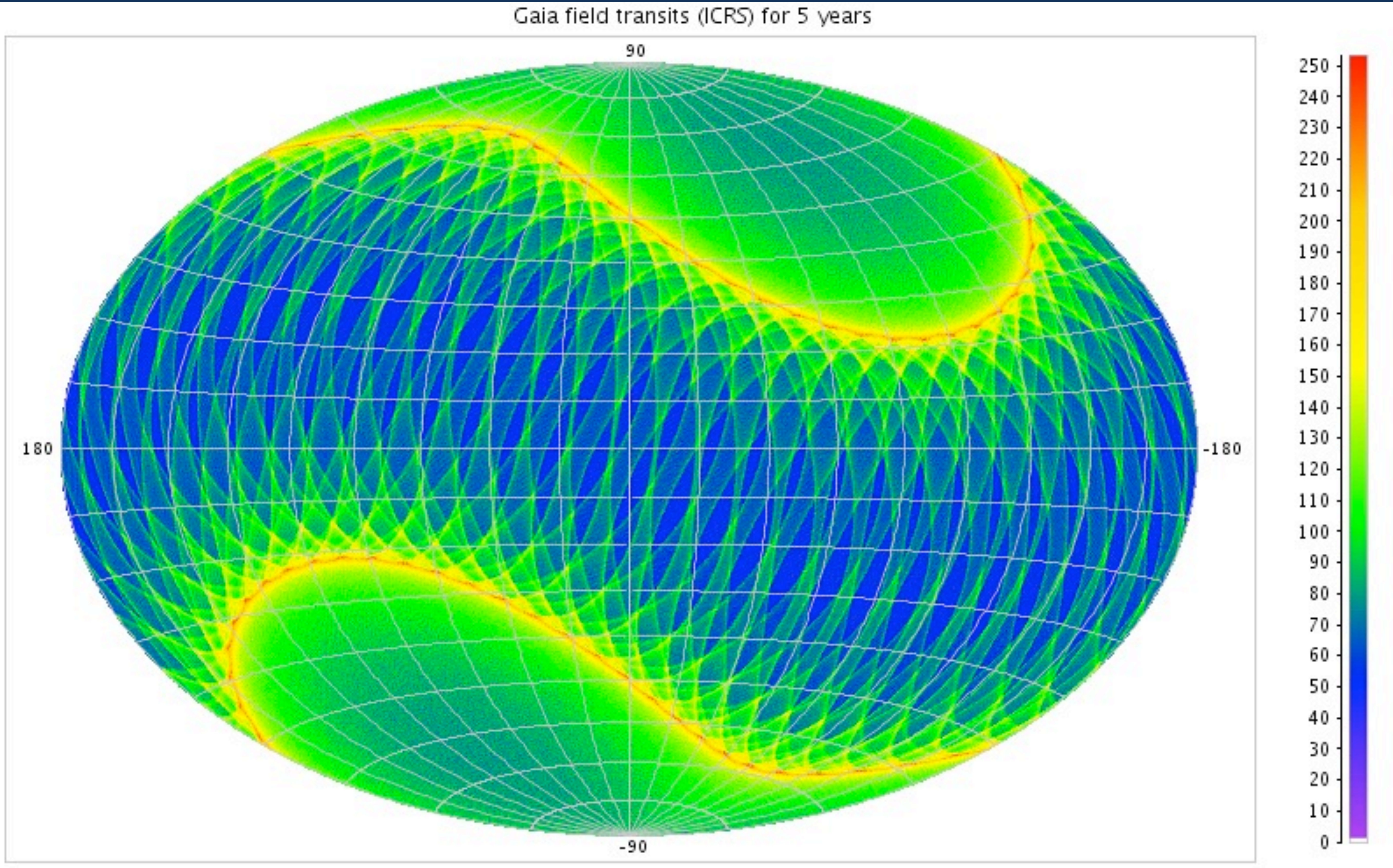


image credit: Lennart Lindegren

Nominal Scanning Law



Covariance & correlation

x_i and x_j are estimated values of astrometric parameter i and j.

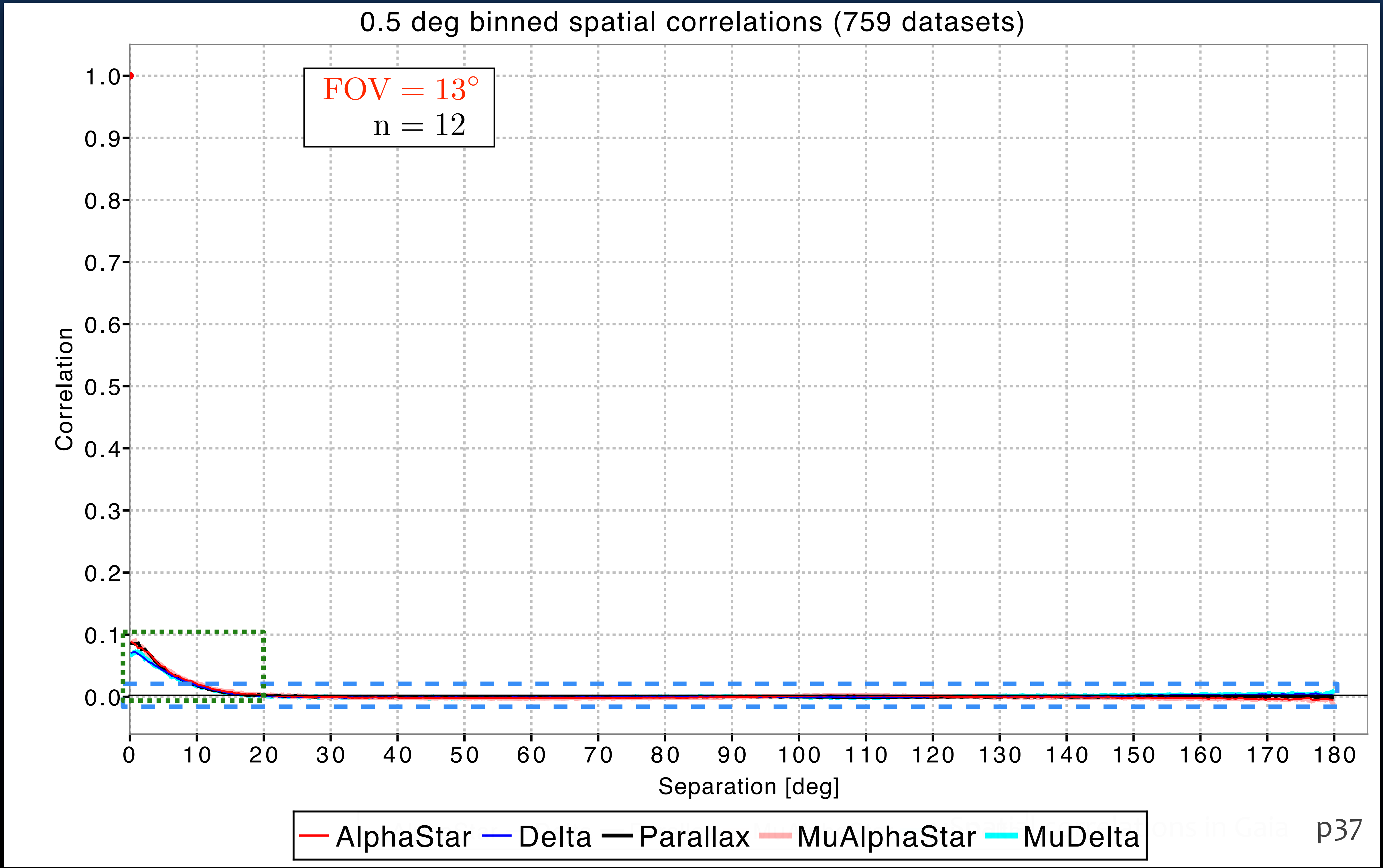
These values are correlated if:

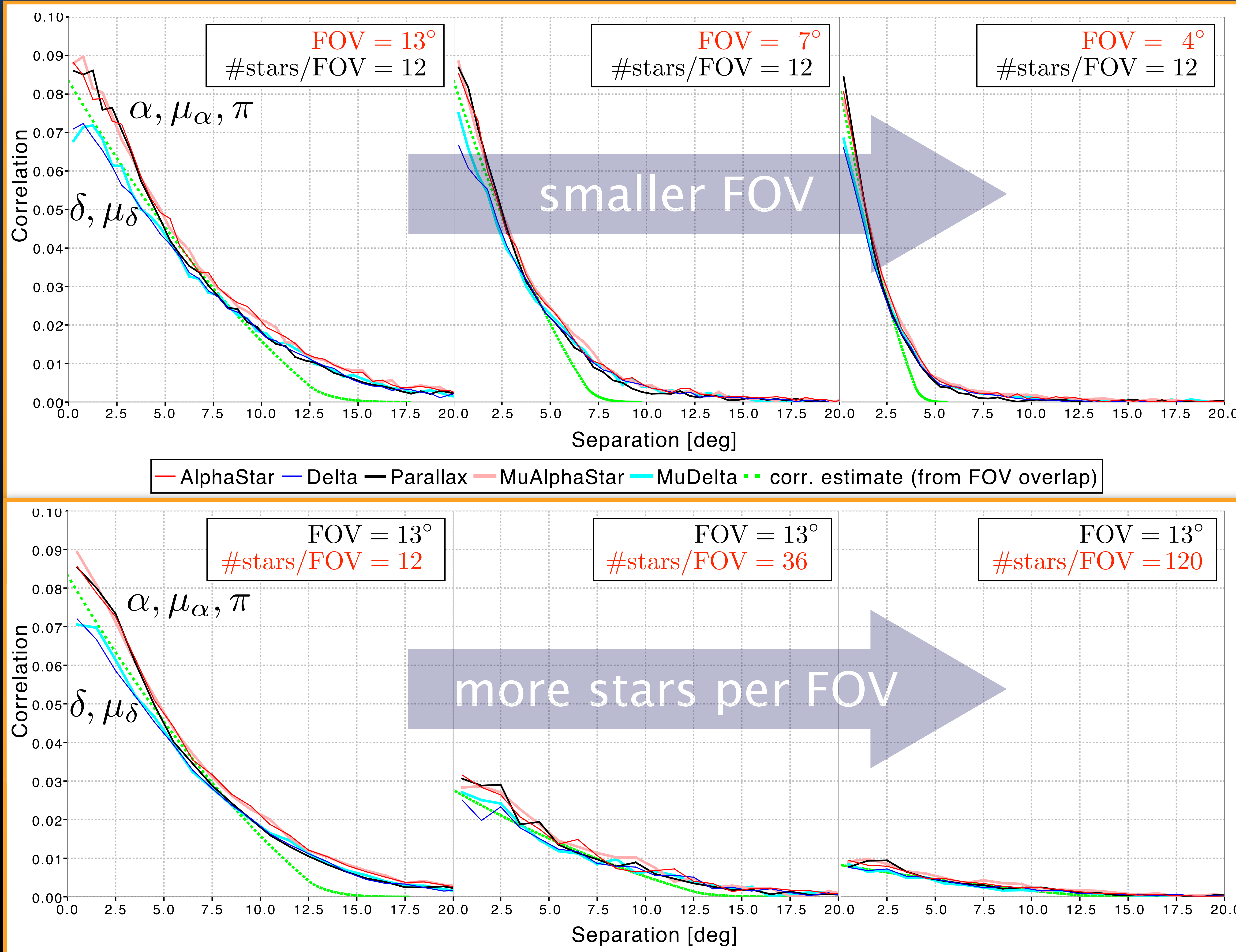
$$\begin{aligned} Cov[x_i, x_j] &= E[e_i e_j] \neq 0 && \text{for } i \neq j \\ e_i &= (x_i - x_{i,true}) && \text{the error} \\ E[e_i] &\equiv 0 && \text{assume no systematic error} \end{aligned}$$

$$Cov[\hat{x}_i, \hat{x}_j] = \rho_{ij}\sigma_i\sigma_j \qquad (= \sigma_i^2 \text{ for } i = j)$$

$$\rho_{ij} = \frac{E[e_i e_j]}{\sqrt{E[e_i^2]E[e_j^2]}} \qquad \text{correlation coefficient}$$

AGISLab results

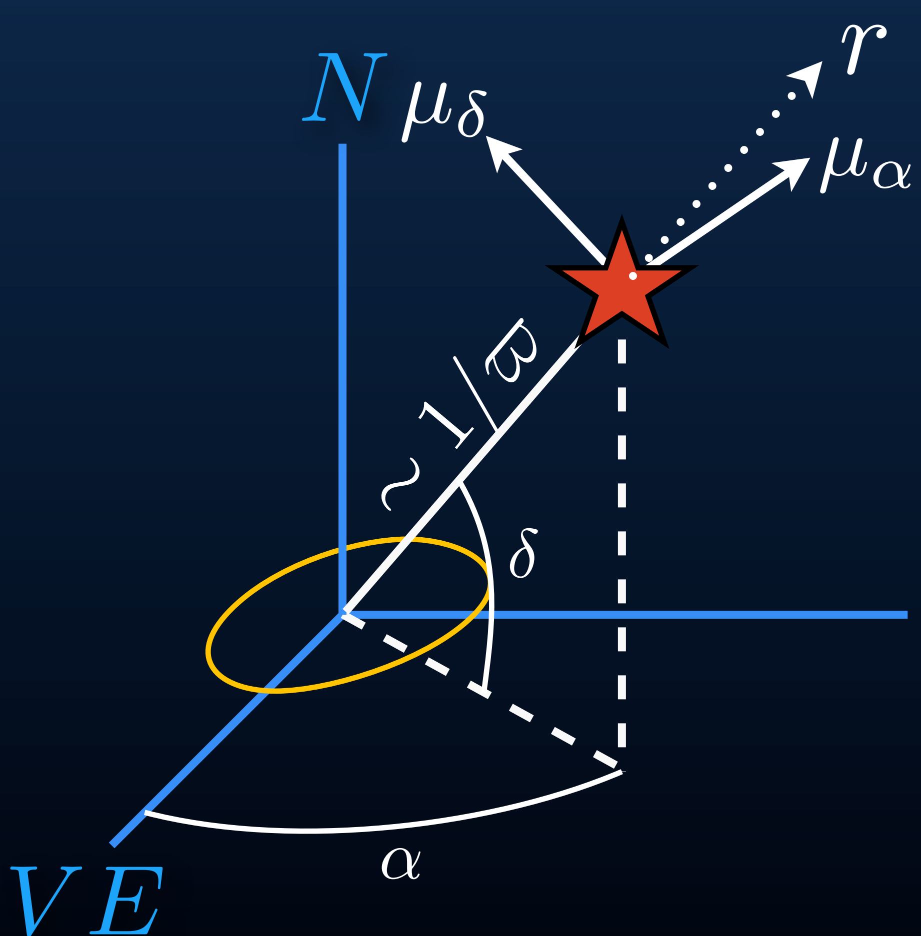




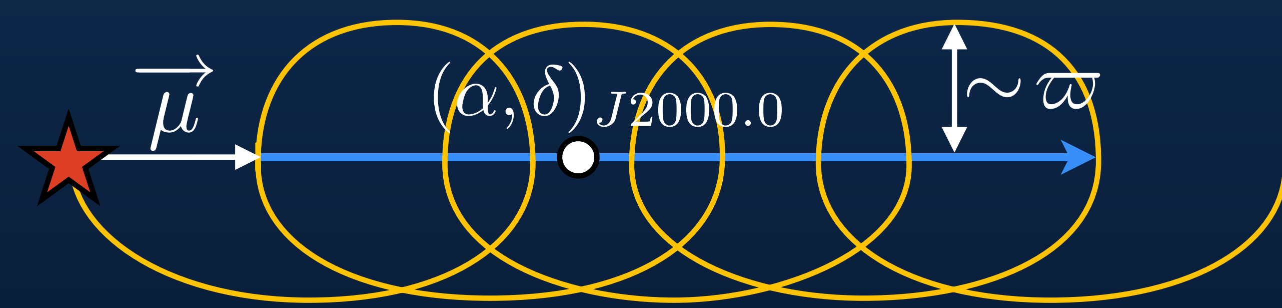
FOV = 13°
#stars/FOV = 12
FOV = 7°
#stars/FOV = 12
FOV = 4°
#stars/FOV = 12
smaller FOV
α, μ_α, π
δ, μ_δ
Correlation
Separation [deg]
AlphaStar
Delta
Parallax
MuAlphaStar
MuDelta
corr. estimate (from FOV overlap)
FOV = 13°
#stars/FOV = 12
FOV = 13°
#stars/FOV = 36
FOV = 13°
#stars/FOV = 120
more stars per FOV
α, μ_α, π
δ, μ_δ
Correlation
Separation [deg]

Astrometric parameters

Which parameters?



Observed position on the sky, measured ~75 times over 5 year:



Only for close-by fast moving stars, radial velocity is apparent: