School Lecture E1: The Sampling Theorem

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Astronomical Imaging

What are we actually measuring?

In orbit a telescope forms diffraction limited images (c. 10mas for a 10m telescope in the visible), but as the wavefront passes through turbulent layers of the atmosphere it is distorted; the result is that the image quality is degraded to a FWHM of around an arcsecond. Locally we may treat this as a convolution by a Point Spread Function $\phi_{\rm atmos}$:

$$I_c = I_{\rm orbit} \otimes \phi_{\rm atmos}$$

What are we actually measuring?

- The (R, G, B) values are related to the intensity recorded by a CCD
- The image I_c delivered by the telescope is continuous, but we have a finite number of finite-sized pixels; i.e.

$$I_j \equiv I(x_j) = \int_{x_j-\frac{1}{2}}^{x_j+\frac{1}{2}} I_c(y)\,dy = \int_{-\infty}^{\infty} P(y-x_j) I_c(y)\,dy$$

where

$$P(x) = \begin{cases} 1 & |x| < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

- We can only measure $I(x)$ at the points x_j, but it is defined everywhere.

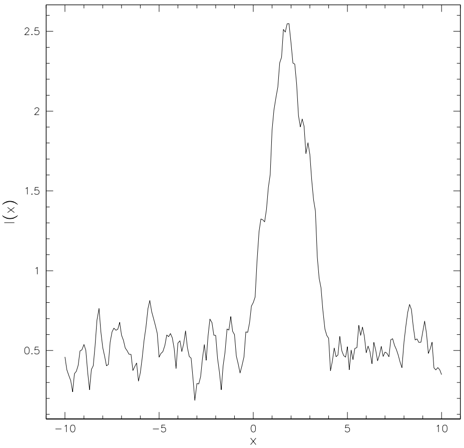
What are we actually measuring?

So

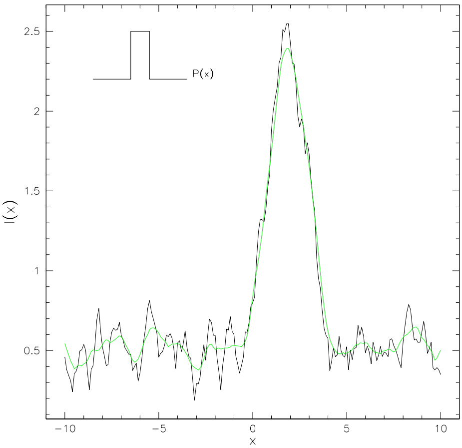
$$\begin{aligned} I &= I_{\rm orbit} \otimes \phi_{\rm atmos} \otimes P \\ &\equiv I_{\rm orbit} \otimes \phi \end{aligned}$$

In otherwords, the PSF includes a contribution from the pixel response function, P.

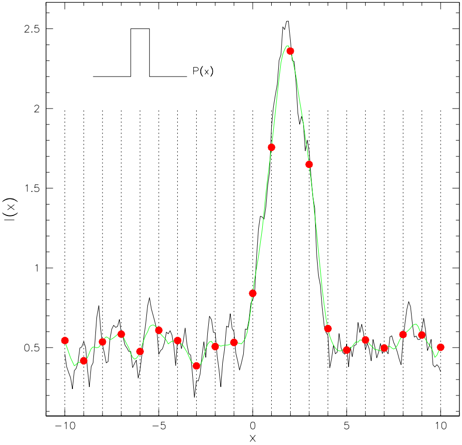
Raw data $I_{\mathrm{orbit}} \otimes \phi_{\mathrm{atmos}}$ at the detector



Pixel-smoothed data



Sampled data



Dirac combs

If we define

$$\delta_T(x) = \sum_{j=\infty}^{\infty} \delta(x - jT)$$

then the function we actually measure (the red dots) is

$$I(x)\delta_1(x)$$

Why is this interesting?
Because it turns out that, under some conditions, we've lost no information by only sampling at the pixel centres.

Fourier Transforms

If we define

$$\begin{aligned} F(f) &\equiv \int_{-\infty}^{\infty} F(x) e^{2\pi i f x}\, dx \\ F(x) &\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} F(f) e^{-2\pi i f x}\, dx \end{aligned}$$

then

$$\delta_T(x) \leftrightarrow \delta_{1/T}(f)$$

Convolutions

Let us agree to use $\mathcal{F}$ to denote a Fourier transform:

$$\mathcal{F}(F(x)) \equiv F(f)$$

You will remember that

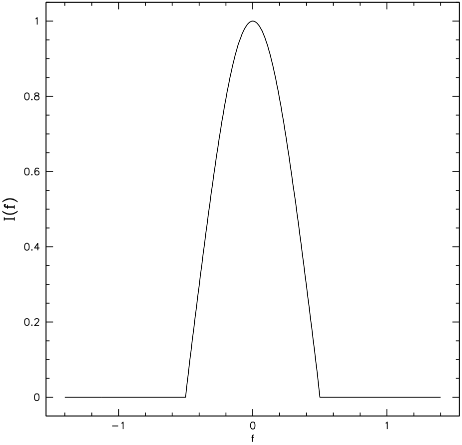
$$\mathcal{F}\left(\int_{-\infty}^{\infty} G(y)H(x-y)\,dy \equiv G \otimes H\right) = G(f)H(f)$$

So

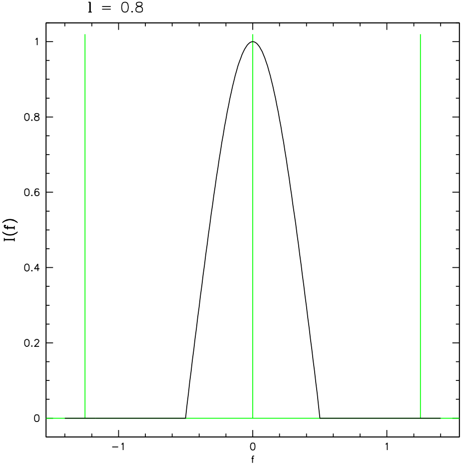
$$\mathcal{F}\left(I(x)\delta_l(x)\right) = I(f) \otimes \delta_{1/l}(f)$$

Let us assume that I is *band-limited*; i.e. that its Fourier transform vanishes outside some interval (which I shall take to be $-1/2...1/2$ for now).
We just wrote:

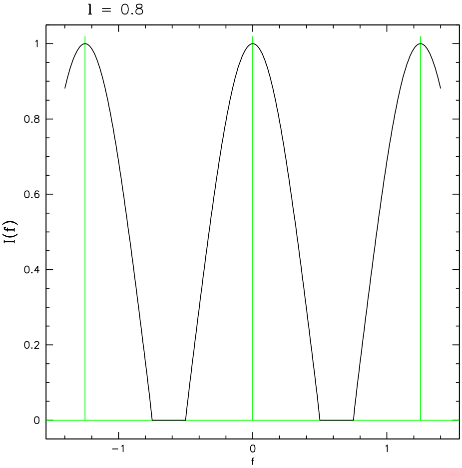
$I(f)$



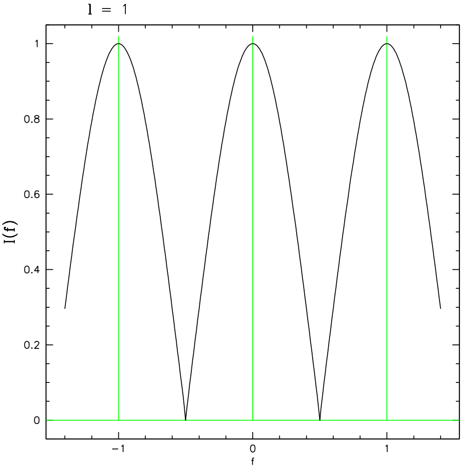
$I(f)$ and δ_1



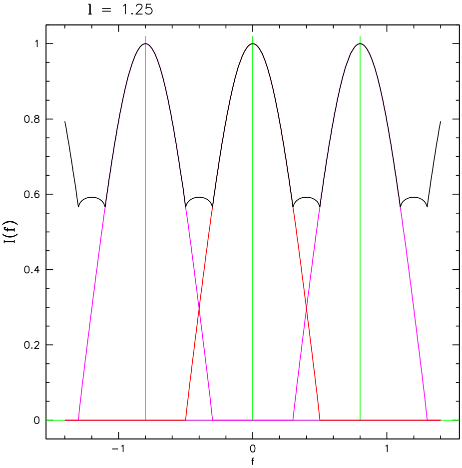
$I(f) \otimes \delta_1$



$I(f) \otimes \delta_1$



$I(f) \otimes \delta_1$



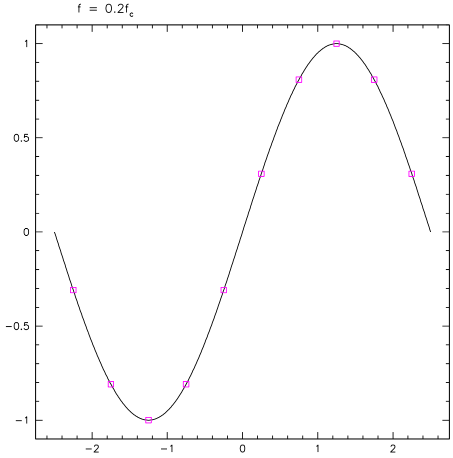
Nyquist Sampling

Something happened at $l = 1$, or (more generally) at

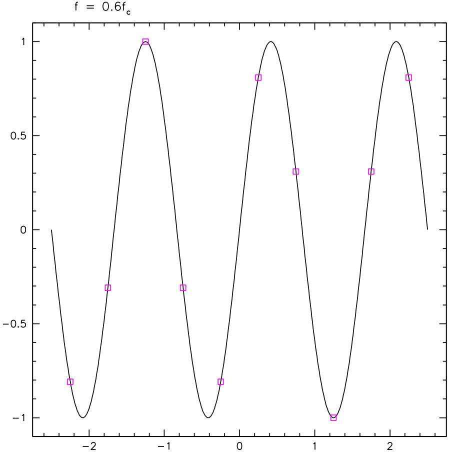
$$l = \frac{1}{2f_c}$$

Let us look at a set of independent sinusoids (*i.e.* Fourier modes) with a variety of frequencies, and try to find out what happened at $l = 1$.

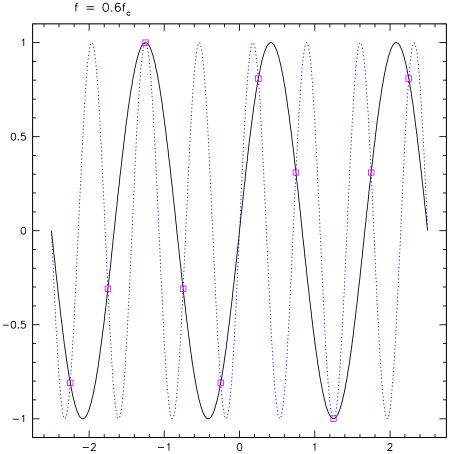
$\nu = 0.2 f_c$



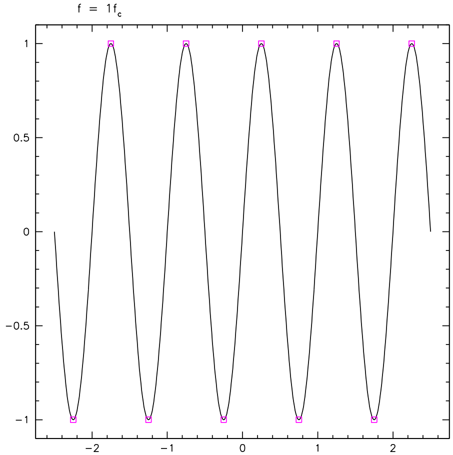
$\nu = 0.6 f_c$



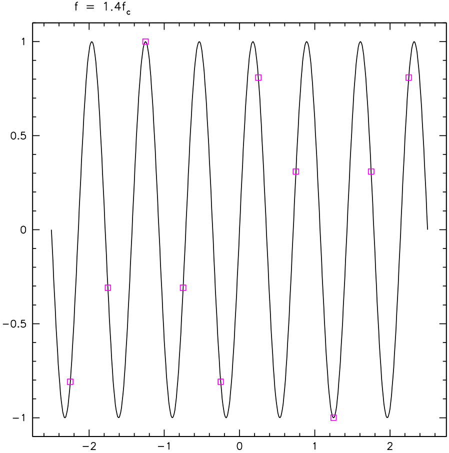
$\nu = 0.6 f_c$



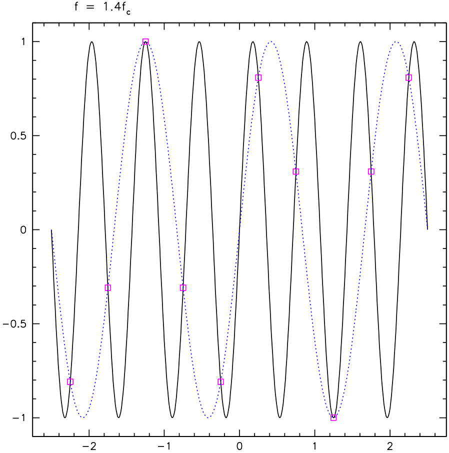
$\nu = 1.0 f_c$



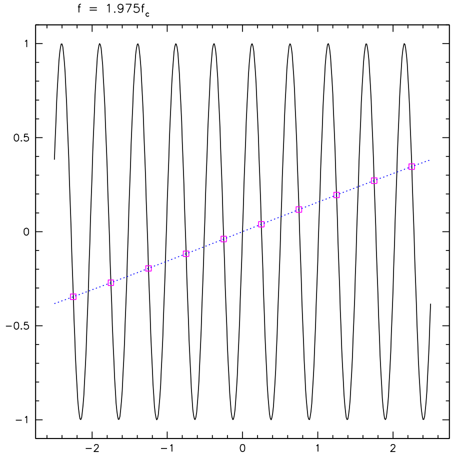
$\nu = 1.4f_c$



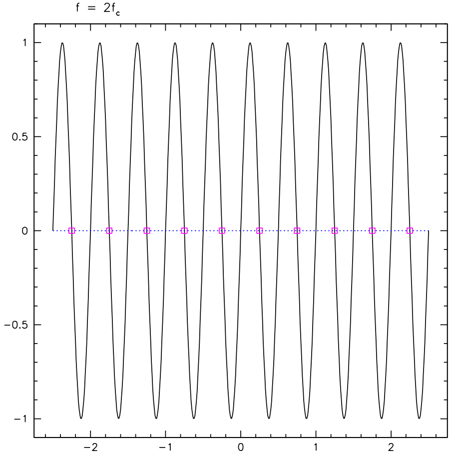
$\nu = 1.4 f_c$



$\nu = 1.975 f_c$



$\nu = 2.0 f_c$



Aliasing

I.e. we can draw more than one curve through a given set of points — any Fourier component above f_c is indistinguishable from a component in $[-f_c, f_c)$.
This is known as aliasing.

Is being band-limited trivial?

The wave front from a distant star is essentially a plane wave; and ideal telescope will convert this into a point:

$$e^{ik.(x-x_0)} \longrightarrow \delta(x - x_0)$$

In reality we don't have infinitely large telescopes, so the incoming wave is

$$\begin{array}{rl} e^{ik.x} & r \leq R \\ 0 & r > R \end{array}$$

so

$$\phi \propto \int_{r \leq R} e^{ik.x}\, dx$$

So yes, at $2R/\lambda$ — but this is irrelevant in most cases; 1 arcsec for an $R = 5cm$ telescope.
In reality, we have an atmosphere, and

$$\phi \sim \mathcal{F}^{-1}\left(e^{-k^{5/3}}\right)$$

Springs

Consider a set of point masses m connected with springs of length d and with spring constants κ.
If you calculate the dispersion relation you'll easily find

$$\omega^2 = \frac{4\kappa}{m} \sin^2\left(\frac{kd}{2}\right)$$

i.e. waves with frequencies $\omega > 4\kappa/m$ are exponentially damped; the transition occurs at a wavenumber of $k = d/2$.
Lord Kelvin knew that materials are opaque to light with $\lambda \lesssim 400$ nm; but he knew that $d \ll 200$ nm.

> *"I believe that by imagining each molecule to be loaded in a certain definite way by elastic connection with heavier matter [I can explain the discrepancy] ... it is not seventeen hours since I saw the possibility of this explanation"*

Sinc Resampling

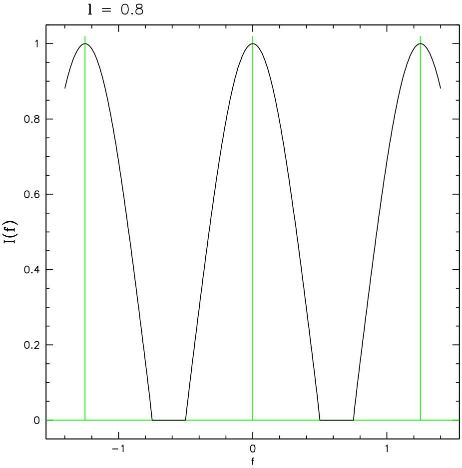
Our CCD measured

$$I(x)\delta_1(x)$$

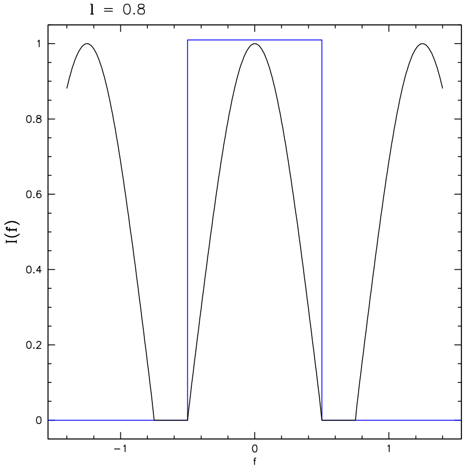
and you know that

$$\mathcal{F}\left(I(x)\delta_l(x)\right) = I(f) \otimes \delta_{1/l}(f)$$

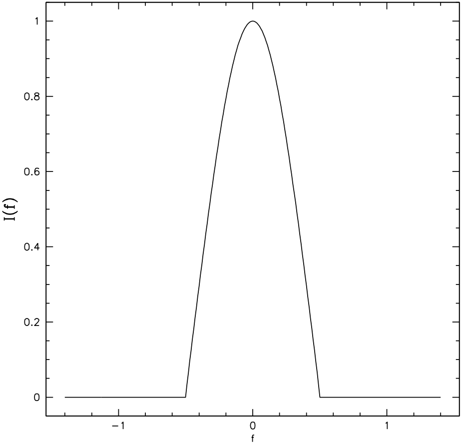
$I(f) \otimes \delta_{1/T}$



$\left(I(f) \otimes \delta_{1/T}\right) \times P$



$I(f)$



Sinc Resampling

In symbols,

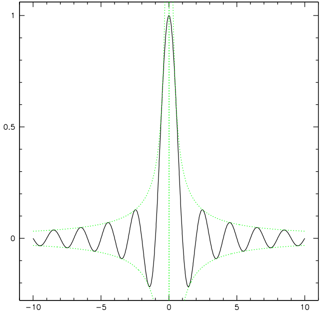
$$\mathcal{F}\left(I(x)\delta_l(x)\right) = I(f) \otimes \delta_{1/l}(f)$$

$$\mathcal{F}\left(I(x)\right) = \left(I(f) \otimes \delta_{1/l}(f)\right) \times P(x)$$

i.e.

$$I(x) = \left(I(x) \times \delta_l(x)\right) \otimes P(x)$$

$$I(x) = \sum_j I(x_j) \frac{\sin\left(\pi(x_j - x)\right)}{\pi(x_j - x)} \equiv \sum_j I(x_j)\,\mathrm{sinc}(x_j - x)$$



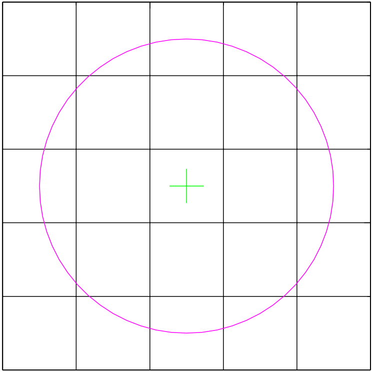
Why should you care?

- Aperture fluxes

  If you want to know

$$F \equiv \int_{r<a} w(r) I(\mathbf{x})\, 2\pi r\, dr$$

  How should you evaluate it?



Aperture fluxes

Maybe we should weight each pixel with its area within the magenta circle, multiplied by w evaluated at the centre of the pixel?

$$F \approx \sum_{ij} I_{ij} w_{ij}$$

We can do better than that.

$$\begin{aligned}
F &\equiv \int_{r<a} w(r) I(\mathbf{x})\, 2\pi r\, dr \\
&= \int_{r<a} w(r) \sum_{ij} I_{ij}\, \mathrm{sinc}(x_i - x)\, \mathrm{sinc}(y_i - y)\, dx\, dy \\
&= \sum_{ij} I_{ij} \int_{r<a} w(r)\, \mathrm{sinc}(x_i - x)\, \mathrm{sinc}(y_i - y)\, dx\, dy \\
&= \sum_{ij} I_{ij} w_{ij}
\end{aligned}$$

N.b. the weights w_i are independent of the data values, and can be evaluated once and for all.

Image Resampling

If we have two images taken with the telescope pointing in slightly different directions, how should I align them?

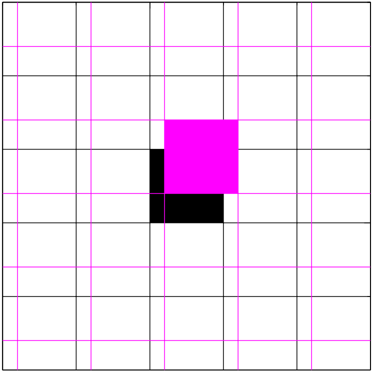


Image Resampling

Maybe we should weight each input pixel with its overlap with the tasteful magenta pixel? We can do better than that. If the black and magenta pixels have identical shape and sensitivity profile, then all we've done is evaluate

$$I(x) = \int_{-\infty}^{\infty} P(y - x) I_c(y)\, dy \equiv P \otimes I_c$$

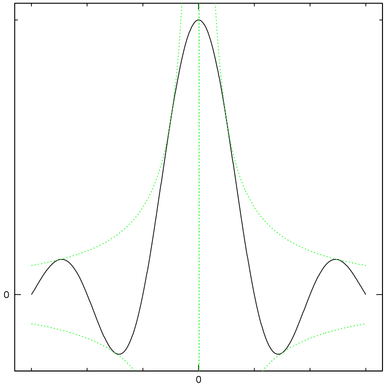
at a different point. But, because we know that I_c is band-limited, we can evaluate it as

$$I(x) = \sum_{ij} I(x_i, y_j)\, \mathrm{sinc}(x_i - x)\, \mathrm{sinc}(y_j - y)$$

In reality, the telescope scale is probably slightly different, and the camera's probably rotated. It's almost always sufficient to correct the flux by the Jacobian of the coordinate transformation.

Lanczos Kernels

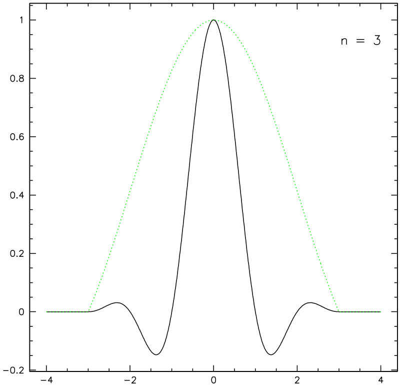
One problem with the sinc kernel is that it has very non-local support:



Lanczos Kernels

A popular modification of the sinc kernel is a *Lanczos(n)* kernel,

$$L_n(x) = \begin{cases} \mathrm{sinc}(x) \times \mathrm{sinc}(x/n) & |x| \leq n \\ 0 & \text{otherwise} \end{cases}$$



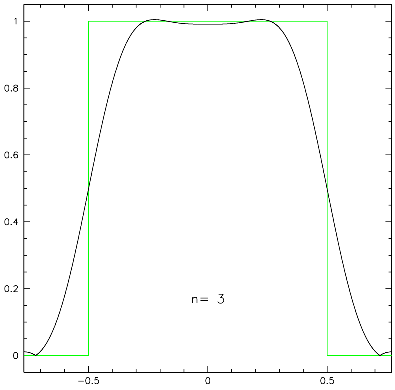
Lanczos Kernels

What does this do in Fourier space?

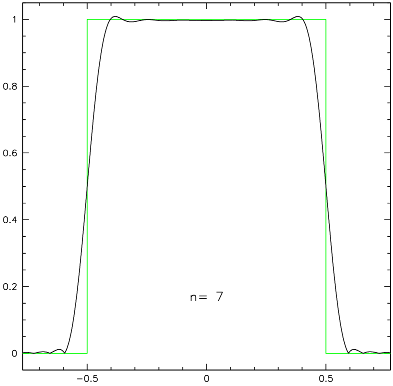
$$\mathrm{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}$$

$$\begin{aligned} L_n(x) &= \mathrm{sinc}(x) \times \mathrm{sinc}(x/n) \times P(n) \\ L_n(k) &= \mathrm{sinc}(k) \otimes \mathrm{sinc}(kn) \otimes P(kn) \\ &= P(k) \otimes P(k/n) \otimes \mathrm{sinc}(nk) \end{aligned}$$

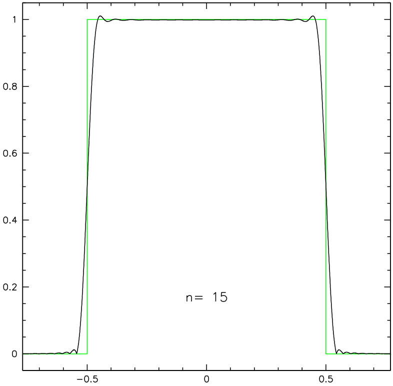
$n = 3$



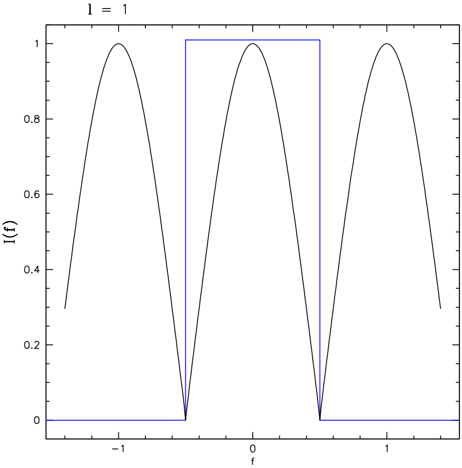
$n = 7$



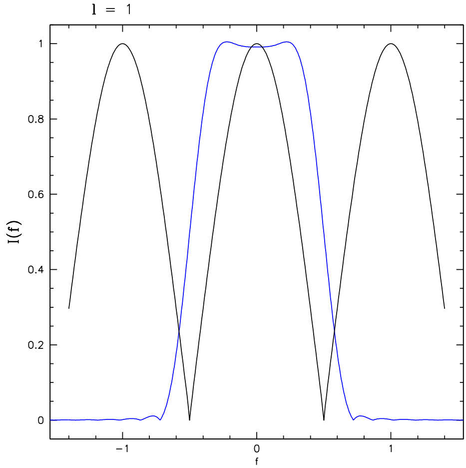
$n = 15$



$(I(f) \otimes \delta_{1/T}) \times P$ Critical sampling



$(I(f) \otimes \delta_{1/T}) \times L_3$ Critical sampling



$(I(f) \otimes \delta_{1/T}) \times L_3$ 20% oversampled